Student Difficulties in Abstracting Angle Concepts From Physical Activities with Concrete Materials

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A Year 3 angles unit using the Teaching for Abstraction paradigm was developed on the basis of earlier empirical research and trialed by 12 NSW teachers. Analysis of data submitted by the teachers showed that the unit was generally successful but a number of minor areas for improvement were identified. An analysis of independent classroom observations of students working on the angles lessons is reported in the present paper. It was found that the difficulties students experienced could be classified as matching, measuring, drawing, and describing errors. The results clearly support the hypothesised steps in learning by abstraction.

The angle concept is exceptional because it arises in so many different contexts. For example, angles are not only used to describe the shape of the corner of a geometrical figure but they are also used to specify a direction, an amount of turning or opening, and an inclination or slope. Mitchelmore and White (2000a) showed how the multifaceted nature of the angle concept makes it a difficult concept for children to learn, and proposed a theory of learning angle concepts by successive abstraction and generalisation. They proposed that young children initially recognise superficial similarities between physical situations and abstract everyday concepts such as table corner, road junction, door, and roof. Gradually, they recognise deeper similarities between these objects and abstract more general concepts such as corner, turn, and slope. They may be said to have formed a general abstract angle concept only when they see that all these concepts in turn have an even deeper similarity.

Mitchelmore & White (2000a) showed that children can identify so-called *2-line angles* (e.g., corners of a room, road intersections, pairs of scissors), where both arms of the angle are visible, as early as Year 2. On the other hand, although young children seem to understand very well a number of *1-line angle* situations (doors, windscreen wipers, ramps, etc.) and *0-line angle* situations (e.g., doorknobs, pirouettes, wheels), even by Year 8 many students cannot interpret these situations using angles. To do so, students have to recognise that these objects are similar in that they can all be regarded as consisting of two linear parts that meet at a point and that the relative inclination of these two parts has a crucial significance. For 1- and 0-line angle situations, one or both linear parts have to be imagined or remembered.

The same researchers (Mitchelmore and White, 2000b) proposed that the teaching of angle should consist of guiding students to make the abstractions and generalisations just described. They advanced three principles of their method, which they called *Teaching for Abstraction*:

- *Familiarity.* Students should first become familiar with a variety of angle contexts.
- *Similarity.* Teaching should then focus on helping students recognise the similarities between these contexts.

B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 583-591). Sydney: MERGA. ©2002 MERGA Inc.

- *Reification.* Activities should be undertaken whereby the recognised similarity becomes abstracted to an angle concept which can be operated on in its own right.

The present paper describes an attempt to put this theory into practice, focussing on the difficulties which students must overcome.

A Year 3 Angles Unit

Following exploratory research using one-to-one and small-group teaching experiments (White & Mitchelmore, 2001a, 2001b), an experimental Year 3 unit covering the angles material in the 2D Space strand of the K-6 mathematics syllabus was developed for the NSW Department of Education and Training and trialed during 2001.

Content

The unit (Mitchelmore & White, 2001a) consisted of 10 lessons, as summarised in Table 1. The first 5 lessons dealt with 2-line angles and the second 5 lessons dealt with 1-line angles. It was judged that 0-line angles would be too difficult for Year 3.

Table 1
The Content of the Angles Unit

Lesson	Lesson
1. **Pattern block corners** Matching and comparing pattern block corners, introducing the word "angle"	6. **Clocks** How a clock uses angles to show the time
2. **More corners** Corners in the classroom, right angles	7. **Doors** Interpreting the opening of a door in terms of angles
3. **Comparing corners** Making an angle tester to informally measure corners	8. **Slopes** Interpreting slope in terms of angles
4. **Scissors** Using angles to describe the amount of opening of a pair of scissors	9. **Angles, angles everywhere** Matching angles across all the situations investigated in this unit
5. **Scissors-like objects** Measuring and matching angles in other objects with two parts connected by a pivot	10. **Creating angles** An open-ended task involving a variety of different angles

The unit followed the principles of Teaching for Abstraction in the following ways.

- *Familiarity.* Students explored a variety of angle contexts (corners, scissors, clocks, doors, slopes) separately to learn about their crucial, angle-related features.
- *Similarity.* Teaching employed frequent direct and indirect matching of angles:
  - In *direct matching*, the angle in one context was physically superimposed upon the angle in the other context. For example, the corner of a pattern block was fitted into the corner of a window frame.

- In *indirect matching*, an intermediate angle-like object was used to indicate how the angles in the two contexts were similar to each other. For example, a bent straw was used to show that many angles in the classroom were "the same".

Matching was particularly crucial in the case of 1-line angles. The approach used was to help children see how to interpret each position in terms of the angle formed between the object and some "neutral" position (12 o'clock on the clock, the closed position of the door, and the horizontal direction for a slope).

- *Reification.* Several activities were aimed towards abstracting the angle concept from the concrete situations which were seen to be similar:
  - Angles were *measured* by finding how many "little angles" (30°) fitted into them. The "angle tester" shown in Figure 1 was developed for this purpose.
  - Angles were *drawn* at approximately the correct size and (in the case of clocks and slopes) the correct orientation.
  - Children *described* how different contexts were similar, where the lines and vertex of an angle were, and what an angle was.

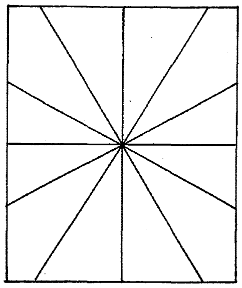

Figure 1: A primitive angle tester made by folding a sheet of paper. Each "little angle" is 30°.

The Trial

Participants. A total of 12 teachers from five DET schools in Sydney trialed the angles unit. The teachers first attended a one-day workshop at which the researchers outlined recent research on student understanding of angle concepts and the teachers worked through many of the student activities and role-played the administration of an assessment interview. They then administered the assessment interview to a target group of eight students, taught the unit at the rate of about one lesson per week, and re-administered the assessment interview to the target students. Finally, the teachers met with the researchers for a one-day de-briefing workshop.

Data collection. Data was collected from two sources.

1. The teachers provided comments on individual lessons and the unit as a whole during the de-briefing workshop. The data from their assessment interviews, together with a folder of work samples for each of the students interviewed, were also collected and analysed.
2. The first author visited each teacher twice. During each visit, she observed one lesson, writing field notes on the flow of the lesson and the behaviour of the target students; interviewed the target students in groups about what they had learned; and

discussed the lesson with the teacher. The first author also viewed videotapes of several student assessment interviews.

Results. An analysis of the data collected from teachers was reported in Mitchelmore and White (2001b). To summarise, teachers' assessments showed that their students' understanding of both 2-line and 1-line angles had improved substantially over the course of the teaching. By the end of the unit, almost all students seemed to have learned to identify and compare 2-line angles, but only about a half of them were able to do so in 1-line angle situations. Teachers felt that the variety of material, the sequential nature of the lessons, and the hands-on nature of the activities were the best features of the unit but indicated several minor areas for improvement.

The teachers' data gave us very little indication of what actually happened in the classroom. It is the purpose of the rest of this paper to provide this extra dimension by analysing the data collected during the class visits. The focus is on the difficulties students experienced in abstracting angle concepts from the various activities they undertook.

Student Difficulties

The lesson field notes, along with transcriptions of the student and teacher interviews and the students' work samples, were first analysed to identify the episodes where students experienced or reported difficulties. (Unfortunately, no examples of Lessons 8 or 10 were observed.) These episodes were then sorted to identify common themes. It was found that the difficulties could be classified into four categories: matching, measuring, drawing, and describing. It may be noted that these four categories of activity are precisely those which were used to implement the similarity and reification principles of Teaching for Abstraction.

Matching

Students experienced few difficulties in matching 2-line angles, either directly or indirectly, when both lines were clearly defined. However, even at the end of the unit, some students could not do so accurately. For example, one student placed the triangular pattern block in a corner as shown in Figure 2 and considered it a match. Another student tried every block until she finally chose the square block as matching the right-angled corner of the desk.

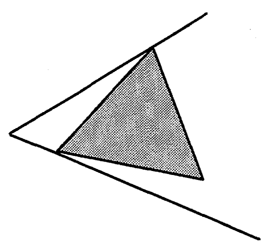

Figure 2. Incorrect matching of two corners.

The scissors caused most students some difficulty. Although the pivot was a focus of the teaching (and the assessment results showed that almost all students identified it as the vertex of the angle formed by a pair of scissors), most students matched the edges of the pattern block with the edges of the blades and therefore did not match the vertex of the pattern block with the pivot.

Many students showed the well-documented error of confusing angle with length or area at some stage during the unit. The angle tester was particularly effective in highlighting the distinctiveness of the angle concept. At the start of Lesson 3, most students thought that the angles at the top and bottom of the rectangle in Figure 1 were greater than those at the sides (either because the areas were greater or the lines longer). By the end of the lesson, as a result of a number of matching activities (for example, fitting a 30° corner of a pattern block into each angle, and folding the angle tester into twelve), most students had realised that the angles were indeed equal. Nevertheless, some students still had difficulty accepting this fact.

As expected from the analysis of student assessments, students had many more difficulties matching 2-line and 1-line angles. For example, many students could open a door to 60° but could not place a triangular block in the door in such a way as to indicate the angle. As a result, many students did not succeed in distinguishing angle from length in 1-line situations. For example, in comparing the openings of two doors, one student said "The big door has the bigger angle" even though the small door was open to a larger angle.

Measuring Angles

Students had little difficulty using the angle tester or pattern blocks to measure the size of 2-line angles (except for the scissors problem mentioned above). Students who had not conceptualised 1-line situations as involving angles experienced the expected difficulties. For example, students who had not made the connection between hours and "little angles" (30°) on the clock found it difficult to check the angles. On several occasions, students suggested checking whether angles on a clock were the same by using a ruler. They seemed to think they were measuring something, so a ruler must be necessary.

Another type of difficulty was revealed in Lesson 6. There were some who measured an angle by counting the lines between the hours. For example, the time between 5 and 10 o'clock was found by counting "6, 7, 8, 9", giving the answer 4 hours. (A similar difficulty is reported in respect to length measurement by Bragg & Outhred, 2000). Some students always counted the hours and/or angles from 12, stating that there were 10 hours between 5 and 10.

Drawing Angles

The many drawing exercises in the unit threw light on a number of difficulties which students experienced. Examples occurred in Lessons 1, 2, and 6.

In Lesson 1, students were asked to make "star patterns" out of pattern blocks by fitting the same corner around a central point, and then to draw the pattern at the centre of each star. Most students successfully made one or more stars, but many could not draw the resulting pattern. Some drawings are shown in Figure 2. Figure 2a shows the angles well. In Figure 2b, the student drew the whole blocks rather than the angles at the centre. Some students could not draw the shape at all (Figure 2c).

In Lesson 2, students found angles in the classroom, matched them with pattern blocks or a bent straw, and then drew the angles. Two groups of students drew an eraser as their example of an angle. They did not mean the angle between the flat sections of the eraser—the angle they drew was the curved part that changes shape as the eraser is used. In the observed classrooms, all the other angles found did consist of two linear parts. However, many students drew the complete objects and often did not indicate the angle at all (Figure 3). Others drew the actual straw in 2 or 3 dimensions, even including the stripes.

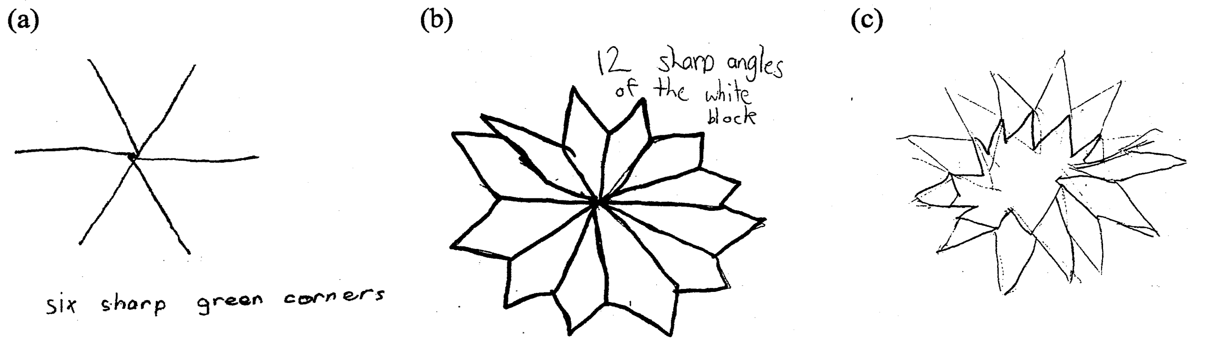

Figure 2. Examples of student work from Lesson 1.

Another type of difficulty was indicated by two students who found an obtuse angle on the blackboard tray, were able to make the angle with a straw, but then drew an acute angle. When it was suggested that perhaps they had not drawn the angle accurately, they correctly matched the straw again but still drew an acute angle.

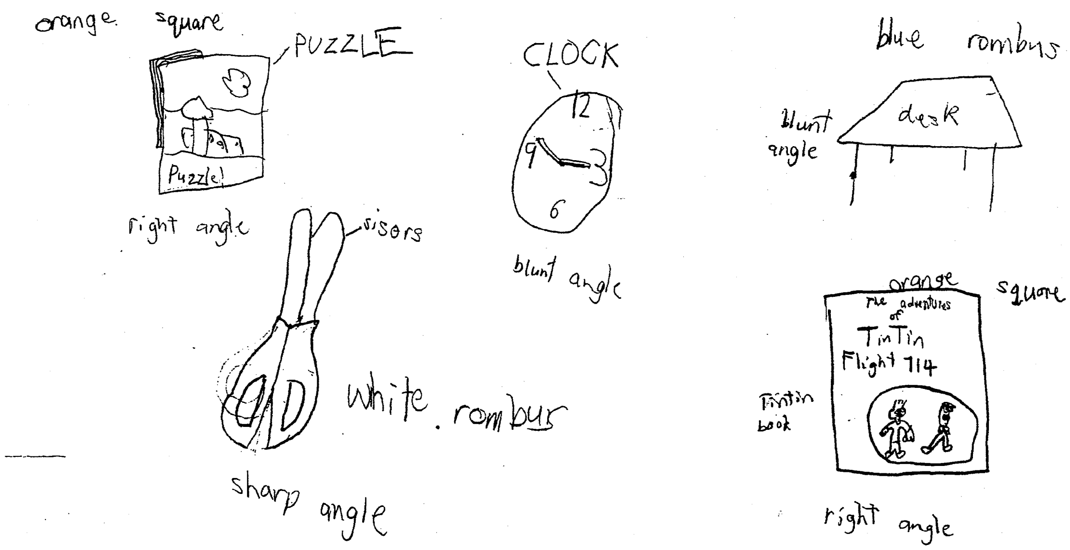

Figure 3. A student's sketch of angles in the classroom.

In Lesson 6, the students made a one-handed clock (using the angle tester for the face and a pencil for the hand) and investigated the angles turned by the hand between various times. On a worksheet, they were given a start and finish time and had to indicate the number of hours and "little angles" from the angle tester, and then draw the angle. Some students drew the same (obtuse) angle for each example, illustrating a similar difficulty to the students who could not copy the blackboard angle in Lesson 2. Other students drew the angles with the correct size, but drew them all from 12 o'clock instead of copying the orientation of the hands. When asked how she had checked that there was a right angle between 1 and 4 o'clock, one student replied:

> I looked at the clock and I knew that 3 was, that 12 to 3 was—makes a right angle. Then I imagine it was 3 o'clock, like that with my fingers, like a L, and that's how I got to do the right angle.

Describing Angles

In sections of Lessons 5 and 9, discussions were planned to focus on the three crucial features of an angle (2 lines, vertex, and opening). Unfortunately, due to pressure of time or because teachers did not appreciate their significance, most teachers omitted these two sections. However, the topic was covered in the post-lesson interviews when students were asked, "What have your learnt?" Here are some typical responses:

S_1: [An angle is] not the outside of lines where the lines spread out, it's where the point is.

S_2: I learned that even though some shapes are different sizes and different shapes that the angles can be the same.

S_3: I learned you can change the thingy, like on the scissors you can change the angle. But on some things like a different one, a cardboard or something like that, you can't.

S_4: If a house door is open a particular angle and the house falls over and the door stays the same, it's the same angle.

Such responses suggest that students were beginning to abstract a general angle concept, but were having difficulties expressing their understanding.

Discussion

The angles unit was clearly effective in teaching students about angles. The analysis of student difficulties shows that student learning could be effectively mapped using the four categories of matching, measuring, drawing, and describing. Since these categories match precisely the pedagogical approaches used in the unit which were intended to promote abstraction, the results lend support to the theory of Teaching for Abstraction and show that it can be used as a practical basis for designing instructional sequences.

Student errors in matching different angle contexts illustrate the crucial significance of matching as a first step in learning to interpret a context using angles. Students who could not match the angle in a new context (e.g., doors) with the angle in an existing context (e.g., pattern blocks) would not have identified in the new context the two lines, the vertex, and the opening which are the essential features of an angle. So they could not be expected to measure, draw, or describe the angles in that context.

Children's drawings show how difficult it is for them to isolate the linear parts of a concrete object which form the arms of the angle. It is clearly a lengthy process to come to regard edges, joins, limbs, and pointers— which may be wide and anything but linear—as lines. Many objects which form the arms of an angle do not *look* like lines at all. They can only come to be regarded as lines when the child recognises the angular similarity between the situation where they occur and a more familiar situation which is known to involve angles and where the lines may be more obvious.

It seems a little easier for students to learn the significance of the vertex and the opening of an angle. The importance of the vertex is most clearly illustrated by children's difficulties in appreciating the significance of the pivot in a pair of scissors. The vertex is more clearly defined in other contexts (even doors) where there is no distraction like the intersection of the two blades on a pair of scissors. In most contexts, the opening aspect is also fairly obvious. Students who do not copy the size of an angle correctly have clearly not yet recognised the significance of the opening. Students who do not copy orientation correctly (e.g., when drawing a clock) may not recognise the significance of the orientation or they may be deliberately transforming the angle to a standard position.

Once students recognise that a context involves angles, they seem to have little difficulty with measuring angle size. Relating angle measurement to the basic concepts of measurement (as is done in *Count Me Into Measurement* [Outhred, Mitchelmore, McPhail, & Gould, in press]) might help those students who count markers instead of spaces. It would not seem a difficult task for students to refine the angle tester (Figure 1) to obtain the standard protractor for more accurate angle measurement.

This analysis of students' difficulties has indicated the important points to emphasise in teaching about angles. In each context, the students must clearly recognise the vertex, lines, and opening aspects of the angle. It is not necessary for them to be able to do so in words—clear verbal definitions seem to follow a long time after understanding. But students should be able to point out the three crucial features and use them in matching. The classroom observations have also confirmed the value of drawing activities in the learning of spatial concepts, both as a way of drawing students' attention to vital features and as a means for the observer to assess students' understanding.

While most students in the present study seem to have abstracted the concept of 2-line angles, and many students abstracted 1-line angles, it is evident that abstracting these concepts is difficult for Year 3 students. The teachers told us that the students needed more time on the angles in each context, and our analysis confirms this need. The unit described in this paper has therefore been revised into one unit of 8 lessons for Year 3 (mainly on 2-line angles) and another unit of 8 lessons for Year 4 (mainly on 1-line angles). The revised angles units will be trialed in 2002.

Acknowledgments

The research reported in this paper was funded by an ACU Large Grant to the third author. We are also grateful for the cooperation of the NSW Department of Education.

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