Arithmetic and Quasi-variables: A Year 2 Lesson to Introduce Algebra in the Early Years

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This paper investigates instruction that assists young children generalise their mathematical thinking in terms of quasi-variables (Fujii & Stephens, 2001). Sixty-five year 2 children participated in a lesson on investigating the patterns in multiples of 5 and 3. From the results it seems that young children are capable of understanding generalising and that specific features of instruction assist this process. The materials, the types of activities and the questions asked by the teacher all play an important role in assisting young children abstract underlying mathematical relationships.

Underpinning all higher levels of mathematics is the ability to operate algebraically (NCTM, 2000). However, in spite of the wealth of international and national research with high school and tertiary students, many students still exhibit a vast array of misconceptions in algebra (TIMSS, 1998) and algebraic thinking still persists as a barrier for participation in high levels of mathematics (NCTM, 2000). Current research is therefore beginning to turn to arithmetic as a key for access to algebra (Carpenter, & Levi, 2000; Carraher, Schliemann, & Brizuela, 2001; Warren & Cooper, 2001; Kaput & Blanton, 2001).

It is believed that the most pressing factor for algebraic reform is the ability of primary teachers to "algebrafy" arithmetic (Kaput & Blanton, 2001), that is, to develop in their students the arithmetic underpinnings of algebra (Warren & Cooper, 2001) and extend these to the beginnings of algebraic reasoning (Carpenter & Franke, 2001). As was argued by Carpenter and Levi (2000), the artificial separation of arithmetic and algebra "deprives children of powerful schemes of thinking in the early grades and makes it more difficult to learn algebra in the later years" (p. 1). The aim of the research reported in this paper is to begin to investigate instruction that helps children take the next steps in generalising and formalising their informal thinking into powerful mathematical ideas that support algebra, transforming arithmetic thinking into algebraic thinking.

Quasi-variable in the early years. Carpenter and Levi (2000) argued that young children could think about arithmetic in ways that not only enhance arithmetic but also provide a foundation for algebra. In particular, they contended that first and second grade children's knowledge of number represented a capability of making and justifying generalisations about the underlying structure and properties of arithmetic. For example, as well as intuitively knowing that 3+7 was the same as 7+3, Carpenter and Levi found that some of the young children showed they understood that this relationship held for all numbers. This ability of children to comprehend general properties of arithmetic in terms of number has been termed by Fuji and Stephens (2001) as "quasi-variable" understanding.

Fuji and Stephens (2001) defined quasi-variables as the use of numbers to indicate an underlying mathematical relationship that remains true no matter what the value of the numbers are. This definition evolved from another part of the study of Carpenter and Levi (2000) in which children were asked whether number sentences such as 78-49+49=78 were

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true or not true. Carpenter and Levi argued that children, who were able to generalise that such number sentences remained true no matter what the first number is as long as you take away and add the same number had an understanding of quasi-variable. Such quasi-variable understanding was evident in answers such as "I believe it's true because you took away the 49 and it's just like getting it back" (Carpenter and Levi, 2000, p 2).

Quasi-variable, therefore, appears to offer a way of expressing generality in arithmetic that can be extended to variable. Of course, care must be taken in interpreting the existence of quasi-variable in children's behaviour because, as Warren (2001) found, children have a tendency to overgeneralise many of the patterns they see in arithmetic, some believing that 2-3 is the same as 3-2.

The focus of this paper. This paper reports on the initial stages of a five-year longitudinal study investigating young children's development of algebraic reasoning. The study focuses on three components of this development: first, the arithmetic underpinnings of algebra (equals, operations and arithmetic properties - Boulton Lewis, Cooper, Atweh, Pillay & Wills, 1998); second, generalisation of arithmetic (leading to quasi-variable and variable understandings); and, third, initial ideas of function (in terms of pattern, relationship and operational change - Warren & Cooper, 2001).

In this paper, we describe a lesson given to three Year 2 classrooms. The lesson was designed to generalise a patterning activity based on multiples of numbers to 10, with the goal of evidencing quasi-variable understanding.

Method

In the first year of the longitudinal study, we are working with three Year 2 classes from an upper middle class state primary school from an inner city suburb of a major city. The sample, therefore, comprised 65 children from three year 2 classes, the three classroom teachers and 2 researchers. These children had only been in Year 2 for a few weeks. They knew about numbers to 10 and had experienced counting to 100, but had not been taught two-digit place value. They had also been introduced to addition and subtraction but not to multiplication.

We worked collaboratively with the teachers to trial teaching ideas and document student learning in relation to teaching actions. The lesson trialled in this paper was an initial activity designed to demonstrate how to generalise arithmetic activities.

Lesson on multiples of 5s and 3s. The aim of the lesson was to identify patterns in multiples of 5 and 3. To do this, the children were provided with a large 100s board, a collection of unifix cubes, small copies of the 100s board on paper, and a coloured pencil.

The lesson began with the large 100s board and the unifix. The children were asked to put their finger beside the 1 on the 100s board and count 5 (1, 2, 3, 4 & 5) and place a unifix cube on the 5. They were then directed to count on another 5 to 10 and place a second unifix cube on the 10. After this, they were directed to keep counting on by 5 and placing unifix until they saw a visual pattern in the placement of cubes. They could then use this cue to complete the pattern.

When their pattern had been checked, the children copied the unifix cube pattern onto a small copy of the 100s board using a coloured pencil. After this, there was a class discussion of particular characteristics that the children had noticed about the 5s pattern. These findings were recorded on the board and used in a class discussion to identify a pattern that would enable numbers to 100 to be in or out of the 5s pattern. Then, the children were asked if they could think of numbers larger than 100 that would be in the 5s pattern. The lesson concluded by repeating the above for 3 – making and discussing the 3s pattern.

Data gathering techniques and procedure. The lesson was taught to each class by one of the researchers. During the teaching phases, the other researcher and classroom teacher acted as participant observers. The lessons occurred sequentially, starting with Classroom 1, Lily's class and finishing with Classroom 3. In each instance the other researcher and classroom teacher recorded field notes of significant events including student-researcher/teacher interactions. The basis of rigour in participant observation is "the careful and conscious linking of the social process of engagement in the field with the technical aspects of data collection and decisions which that linking involves" (Ball, 1997, p. 311). Thus both observers acknowledged the interplay between them as classroom participants and their role in the research process. At the completion of the teaching phase, the researcher and teacher reflected on their field notes, endeavouring to minimise the distortions inherent in this form of data collection and to come to some common perspectives of the instruction that occurred and the thinking exhibited by the children participating in the classroom discussions.

Results

Classroom 1

Due to limited classroom materials, the children in classroom 1 undertook the activity in pairs. They shared a large 100s board and jointly counted and placed the unifix cubes. However, they each made their own copy of the pattern onto a small board. They appeared not to be used to working in pairs and this may have affected their performance.

The children appeared to be reasonably successful at counting and putting on cubes, and most quickly saw the following vertical lines that are the 5s pattern on a 100s board.

1	2	3	4	☐	6	7	8	9	☐
11	12	13	14	☐	16	17	18	19	☐
21	22	23	24	☐	26	27	28	29	☐
31	32	33	34	☐	36	37	38	39	☐
41	42	43	44	☐	46	47	48	49	☐

and so on

They had little trouble copying the pattern onto the paper. However, while some children coloured their pattern using one colour, many copied the pattern using two coloured pencils (a colour for the left-hand column and another colour for the right hand column). When the children were asked to describe the characteristics of the 5s pattern, there was a spread of responses. Many tended to see two columns with each column as a separate pattern, 5 at end of the numbers in the first column and 0 at end of the other. This could explain why some insisted on using two coloured pencils to colour their pattern on the paper cutouts. However, some children did see the 5s pattern as a whole pattern as exemplified by one child's response - "the 5s pattern has numbers ending with 5 or 0.

In order to check the children's understanding of the 5s pattern, they were asked to place their paper copies face down on their desks while the researcher/teacher proposed numbers for the children to indicate if they belonged or not belonged to the 5s pattern. The teacher/researcher began with numbers ending in 0. Most children agreed that numbers ending in 0s belonged to the 5s pattern. The discussion then moved to numbers ending in 5, and finally, to numbers not in the patterns (e.g., 46, 72). To ascertain the children's ability to generalise the pattern, they were asked to give examples of numbers they believed were in the 5s patterns. Most successfully participated in this phase of the lesson.

In order to establish if they could 'see' the pattern in general terms, the children were asked to provide examples of the largest number they felt were in the 5s pattern ("tell us the largest number you can think of that would be in the 5s pattern?"). No one initially gave numbers above 100, so they were specifically asked if there were any numbers greater than 100 in the 5s pattern. This produced the desired result but the numbers were not that large; most gave examples such as 105, 1 005, 1 010. With encouragement, one child gave 1 000 005.

The children were then asked to construct the 3s pattern. They were told to do the same as they did for the 5s pattern but to count in threes until they saw the pattern. They were told to be careful but most children only counted for the first row and then immediately built vertical columns, giving the visual pattern as follows:

1 2 □ 4 5 □ 7 8 □ 10
11 12 □ 14 15 □ 17 18 □ 20
21 22 □ 24 25 □ 27 28 □ 30
31 32 □ 34 35 □ 37 38 □ 40
and so on

The researcher/teacher had to continually reinforce the need to keep counting to get most children to where they had the diagonal pattern as follows:

1 2 □ 4 5 □ 7 8 □ 10
11 □ 13 14 □ 16 17 □ 19 20
□ 22 23 □ 25 26 □ 28 29 □
31 32 □ 34 35 □ 37 38 □ 40
41 □ 43 44 □ 46 47 □ 49 59
and so on

It seemed that for some, the visual of the 5s pattern was so strong that even as they were making the diagonal pattern they tended to regress to a vertical pattern, with their patterns tending to have 'dogleg' shapes.

1 2 □ 4 5 □ 7 8 □ 10
11 □ 13 14 □ 16 17 □ 19 20
□ 22 23 □ 25 26 □ 28 29 □
□ 32 33 □ 35 36 □ 38 □ 40
and so on

The lesson ended with a very brief discussion about what numbers were in the 3's pattern, but no characteristics other than "goes up by 3" and "it's a diagonal" were identified.

Classroom 2

Due to more material being available, each child was given his/her own material and worked alone. Of the three classes, this class appeared to have a greater proportion of low achieving children. From the observations it seemed that overall, this group needed the most assistance in counting, in seeing patterns and in copying patterns.

Compared to the children from classroom 1, the children in this class tended to be slower but more accurate at placing the cubes on the board for the 5s pattern. They also seemed to identify the visual pattern at an earlier stage. However, when they had copied the pattern to the smaller paper board and were asked what they saw, they could not identify any characteristics of the 5s pattern other than it "went up in 5s".

In order to assist them to ascertain the pattern, the researcher/teacher teased out the discussion as follows. First, the children were asked to look at one column at a time, starting with the zero column. They were then instructed to run their fingers down this column and to read out the numbers aloud. However, many children still seemed unable to see the pattern. The process of running a finger down the column and stating numbers aloud was repeated with the focus being placed on the right hand digit (the ones). At this point, most were able to express the pattern as ending in 0s.

In order to test their ability to generalise this component of the pattern, the children were asked to turn their copies of the pattern over while the researcher/teacher proposed 40, 70 and 30 for them to decide if they felt the numbers belonged in the 5s patterns. Nearly all children agreed that they did. They then checked their responses by turning the board back over so that they could see if the numbers were coloured on their paper copy of the 100s board. The same steps were repeated for the second column. Again this process was successful in getting most students to identify the pattern.

After this, the children were asked if they could think of a rule for the 5s pattern with their answers incorporating both columns. The aim of this was to get them to combine the two conclusions above and say that if a number had 0 or 5 for the right digit then it was in the 5s pattern. Only a few children appeared to be able to articulate this response. The combined pattern was reinforced by again asking the children to turn over their paper copy and ascertain if a new set of numbers proposed by the researcher/teacher were in or out of the5s pattern.

In proposing these numbers, it became evident that in order to assist young children generalise the 5s pattern, instruction needed to start with numbers such as 60, 30, 20 and then 35, 65, 75. Care had to be taken with 50 and 55 and, with numbers like 54 where the 5 was at the left hand side rather than the right hand side. It also became apparent when they were asked to identify whether 23 or 32 belonged to the pattern that some children added the numbers and thus believed they did belong as the sum was 5.

When asked for larger numbers, this class mainly gave numbers less than 100 with the exception of a the few examples of larger numbers which consistently had many 5s (e.g., 555).

Compared with classroom 1, the difficulties the classroom 2 children had with the 5s pattern task was mirrored in their inability to generate the 3's pattern. Even after they were told that the 3's pattern was not a vertical pattern, they tended to return to vertical placement of cubes as soon as they were left on their own. Most children followed this sequence:

(1) They counted two numbers and then went vertical, as follows:

1	2	□	4	5	□	7	8	9	10
11	12	□	14	15	□	17	18	19	20
21	22	□	24	25	□	27	28	29	30

and so on

(2) When told to count more, they placed one more unifix and then went

vertical again, as follows:

1 2 □ 4 5 □ 7 8 □ 10
11 12 □ 14 15 □ 17 18 □ 20
21 22 □ 24 25 □ 27 28 □ 30
and so on

(3) When told the vertical was wrong and they had to continue counting, they tended only to do one more row and revert to vertical, as follows:

1 2 □ 4 5 □ 7 8 □ 10
11 □ 13 14 □ 16 17 □ 19 20
21 □ 23 24 □ 26 27 □ 29 30
31 □ 33 34 □ 36 37 □ 39 40
and so on

They only began to see the diagonal pattern with continual reinforcement to "keep counting". The time taken for this meant that there was no time to discuss the pattern.

Classroom 3

This class was a composite class comprising of Year 1 and Year 2 children. We worked with the Year 2 component, consisting of 14 children. They seemed much more able than the preceding two classes. This was confirmed in a later discussion with the classroom teacher.

The children quickly identified the visual pattern in the 5s. They proffered many ideas with respect to the characteristics of the pattern, for example, "2 columns", "5s at end of the first one" and "0s at end of other". They also proposed other patterns for example, "tens going up by 1 as zeros and 5s stay the same". Because of their lack of the place value vocabulary, they tended to talk about the "right hand number" going up in 1's while the left hand number stayed as 0 or 5. At the conclusion to the discussion, the pattern was summarised by more than one student as "any number with 0 or 5 at end".

When asked for larger numbers, the children gave a huge variety of numbers, including unusual ones (e.g., 765) and ones in the millions (e.g., 2 000 0005) and billions ("a billion and 5"). Finally, one child proposed "infinity five", which was immediately followed by another child's proposal "two infinity five".

The classroom 3 children also completed the 3s pattern much more easily than the previous two classrooms; they exhibited few difficulties with pattern completion. Whilst most children tended to move quickly to the pattern as diagonal after one hint to keep counting in threes, a minority exhibited the dog-leg error as follows:

1 2 □ 4 5 □ 7 8 □ 10
11 □ 13 14 □ 16 17 □ 19 20
□ 22 23 □ 25 26 □ 28 29 □
31 32 □ 34 35 □ 37 38 □ 40
41 □ 43 44 □ 46 47 □ 49 59
51 □ 53 54 □ 56 □ 58 59 □
and so on

For some children, the 100s board had to be turned through 45 degrees so that the lines of unifix were vertical before they were comfortable.

In discussing the characteristics of the 3s pattern, there was an interesting collection of proposals. Some children stated that there were 2 numbers between members of 3s pattern, both vertically and horizontally, other children saw the pattern as "going back one" and spoke in terms of changes in the left-hand and right-hand digits (e.g., "for each number in a diagonal, the tens go up by 1 and the ones go down by 1"). One student even proposed the very complex expert pattern, that the digits added to 3, 6 and 9.

Discussion and Conclusions

The role of quasi-variable. The aim of this research was to see if Year 2 children were able to generalise the 5s and 3s patterns and to ascertain if their generalisations was in quasi-variable form. We believe that an understanding which can articulate that any number ending in a 0 or 5 is in the 5s pattern represents an example of using quasi-variables to indicate the underlying mathematical structure of the pattern. However, not all children achieved this level as the responses given by the children within and between the three classes varied. Children in classroom 1 mostly gave examples that contained 10s or 5s in the response (e.g., 105, 1005, 1010 and 1 000 005), whereas the responses in classroom 2 all contained 5s (e.g., 555, 55555). By contrast, Classroom 3 gave a wide variety of examples, clearly indicating an understanding that the most important feature of the 5s pattern is the number in the ones place. We contend that quasi-variable understanding was exhibited in Classroom 3 and not in Classrooms 1 and 2 because even though children in Classrooms 1 & 2 articulated that the 5s pattern has numbers ending in 5 and 0 and the examples they gave of numbers belonging to the 5s pattern were limited. We argue that expressing a generalisation in everyday language is not enough to show quasi-variable understanding; children need to then give evidence of this understanding by offering a wide array of examples, including large numbers beyond the domain that they are exploring. Thus responses such as ∞5 and 2∞5 are crucial for evidence of 'real' generalisation. What is the role of the teacher in assisting children to see generalisations and how much spontaneity is necessary for a child to exhibit quasi-variable understanding? Both these questions need further research.

Classroom instruction. This research has began to delineate some important aspects of instruction that appear to assist children to generalise and formalise their mathematical thinking. First, the overall instruction consisted of a number of phases. The children were ask to complete an activity, reflect on the patterns in the activity and then check their reflections by considering and then posing examples. Then, their thinking was challenged by asking them to give examples beyond the domain of numbers under consideration. This extension to the larger numbers seemed to offer opportunities for insightful discussions about the generalisation. It is conjectured that this discussion offered opportunities for understanding of the quasi-variable to emerge. Second, the detailed instruction in Classroom 2 appeared to assist children to begin to see the 5s pattern. While the instruction did not culminate in generalising the pattern to the extent hoped, the series of small steps helped them to focus on the main features of the task. We contend that such instruction could be classified as 'getting ready' for quasi-variables, teaching strategies that focus children's thinking on the salient features of mathematical structure of the problem.

Third, the integration of visual, auditory and kinaesthetic activities seemed effective. Stating the numbers in the pattern aloud assisted many to focus on the numbers in the pattern rather than on all the numbers on the 100s board. Fourth, breaking the pattern into parts also assisted children to see the pattern as a whole. This was especially effective in

Classroom 2. It was only when they explored the 2 columns separately that they appeared to understand the aim of the activity. Fifth, continual reinforcement such as asking children to turn over their boards and then ascertain whether numbers were in or out of the pattern by putting their hands up seemed to support understanding. The discussion that ensued from questions such as "why is the number in the pattern?" or "why is the number not in the pattern?" also appeared to play an important role. This was especially evident when the classroom discussion focused on whether numbers such as 23 and 32 were examples of the pattern: "If the numbers add to 5 does that mean they are in the pattern? Why? Why not?"

Thinking that supports understanding of generalisation. From these preliminary results, we believe that young children are able to begin to comprehend the structure of mathematics. The use of quasi-variable certainly seems to assist in this comprehension by providing a way for young children to articulate the generalisation. Of course, the performance of children varied; some of the children were quite amazing in the patterns that they could see while others required specific instruction and reinforcement to assist them. As well, the performance of the children posed questions about the nature of generalisation. By its definition, the quasi-variable is numerical, but this activity also contained a strong visual component. The relationship between these two components needs further investigation. How does the visual pattern support or mitigate against the articulation of the numerical pattern? It seems that the visual pattern certainly did not help in some instances, especially with the transition from the 5s pattern to the 3s pattern. Many children experienced little difficulty in integrating the visual with the numerical. In fact the visual seemed to assist them in quickly identifying the pattern and expressing the pattern in a numerical form. However, the vertical pattern of the 5s seemed to make the diagonal pattern of the 3s difficult to see. In fact, in order to complete the diagonal pattern of the 3s some children needed to turn their board through 45^0, thus converting the pattern to the vertical. The role of kineasethic reinforcement also needs further investigation. In this instance running their fingers down the numbers and saying the numbers seemed to assist in breaching the gap between the visual and number pattern.

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