Pizza for Dinner: “How Much” or “How Many”?

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This paper engages current thinking about the links between context and learning. It reports on a study of students’ understandings of sharing and part/whole distributions within the family pizza dinner context, represented by a circular region model. Central to the analysis is the part that informal knowledge plays in the development of rational number understanding. The investigation reveals that for some students whole number operations play a more significant role in the pizza context than do conceptualisations of part/whole distributions. The suggestion is that rational number development requires a gradual pedagogical exposure to a range of structural representations which embody the concept of part/whole.

Changes to instructional classroom practices have been apparent in school mathematics curricula in Australasia for some time. The most recent shift which acknowledges a local, subjective and socially constructed world offers a view of knowledge which no longer draws its inspiration from the psychological experience of a stable world, but rather takes as its central tenet the idea that knowledge evolves with community and culture. As a result mathematics is reconstituted as useful, relevant and meaningful. Usefulness, relevance and meaningfulness, by design, involve context. Contextual learning is said to support better understanding for students and ultimately better opportunities for those same students.

The term c*ontextual knowledge* has come to refer to a wide range of philosophies and curriculum interventions on mathematics education over recent years. Theoretical positions which have attempted to link cognition with contexts of social experience are many and varied, each providing a model of learning which acknowledges the active role of the learner. At the same time, each assigns differential importance to the contextual social processes and to the individual processes of learning. For example, context plays a part in cognitive development in constructivist epistemologies of knowledge acquisition. However in these theories more attention is given to individual cognitive structures than to social processes. Learning is viewed as the individual mind being *influenced* by the social world. Theories of social practice develop their ideas of social processes differently. These situated theories of learning draw on an anthropological realisation that cognitive abilities and capacities are formed and constructed *within* social phenomena. Knowledge is participatory, distributed and social situated (Rogoff, 1990; Sfard, 1998).

Subtle in essence, the shift has enabled new possibilities for investigations into learning and knowledge production. We draw on analyses of everyday social practices, in which knowledge is held to be an integral part of the specific activity, context, and culture in which it is located (e.g., Forman, 1996; Greeno, 1997; Lave, 1988; Nunes, Schliemann & Carraher, 1993; Saxe, 1991), for our exploration into rational number. We asked: what part does context play in the development of fraction understandings? Early understandings of fractions can be described as a process of initiation into a social understanding of when and how to act in particular familiar situations (Cobb & Yackel, 1996). Integral to that process is the real-life circumstantial knowledge that goes under the name of *intuitive knowledge* (Leinhardt, 1988; Resnick & Singer, 1993), *situated knowledge* (Brown, Collins & Duguid, 1989), and *informal knowledge* (Ginsburg, 1982; Saxe, 1988). Those understandings might be drawn upon by the student in response to problems posed in the context of real life familiar situations.

Many researchers, however, claim that informal fraction knowing is limiting, more reflective of whole number knowledge than of the characteristic principles of rational number. Hart (1988), and Lesh and colleagues (1987) maintain the students' informal knowledge of rational number is routinely misconceived. Many students draw on their informal strategies of partitioning and ratio and come to regard individual parts of a partition as discrete objects (see Mack, 1990). Thinking in terms of "how many?" they fail to make the critical transition to thinking of rational numbers as also a representation of "how much?" Likewise, in ratio problems students tend to focus on the individual parts, treating the numerator and denominator of a fraction as separate numbers rather than recognising that the rational number itself is a number (see Resnick & Singer, 1993; Lamon, 1993).

Knowing about formal school fraction knowledge entails entering into and "participating in a community of people who practice mathematics" (Hiebert et al., 1996, p.248). Many researchers, however, believe that students' informal whole number knowledge can interact positively with the instructed knowledge of the curriculum. Mack (1993) and Streefland (1993) both describe how young children are able to solve a variety of problems involving partitioning or sharing, and Resnick and Singer (1993) report on the insightful invented solutions to ratio problems provided by students. Informal knowledge becomes a springboard for formal school fraction knowing and later higher level mathematical understanding, provided students are given realistic problems and contexts that do not emphasise symbolic manipulations.

Research Study

This paper reports on data drawn from New Zealand's National Education Monitoring Project (NEMP). In the 2001 NEMP study, 2869 students from 254 schools were assessed on their content knowledge and process skills in mathematics (Crooks & Flockton 2002). Approximately half of the students were from Year 4 (ages 8–9) and half were drawn from Year 8 (ages 12–13). The project reported here focuses on students' knowledge and explanations of fractions.

The researchers viewed videotapes of 60 Year 4 and 50 Year 8 students, randomly selected from the NEMP bank of student responses. The particular aim was to describe, analyse, and discuss differential ability amongst the 160 students to recognise, identify, name and explain part-whole fraction representations and from those responses, to identify the role which context plays in the development of fraction understanding. What is at stake are the understandings students have at Year 4 and at Year 8 of rational number, and their capacity to explain those understandings in everyday language.

Both researchers viewed all of the tapes of students' involvement. Each student was introduced to the research by the interviewer's words: "Here are two whole pizzas for a family dinner. This one [points] is a pepperoni pizza and this one is a ham and pineapple pizza [points]. After dinner, some of each pizza was left over" [removes sections]. The students were then asked the following questions, and provided with as much time as needed to respond:

- How much of the pepperoni is left?
  Prompt (if answer not given as fraction): What fraction or part is left?
- How much of the ham and pineapple is left?
  Prompt (if answer not given as fraction): What fraction or part is left?
- Altogether, how much pizza is left?
- Now we are going to think about two different ways of using up the pizza that is left over. If four children had a quarter pizza each, then how much would be left?
  Prompt: You can move the pieces of pizza around to help you work it out.
- Year 8 students were asked the following additional questions:
- This time imagine that the two of us are going to have an equal share of the pizza that is left. What fraction or part of a whole pizza do we each get?
  Prompt: You can move the pieces of pizza around to help you work it out.
- Can you explain to me how you worked that out?

Two model pizzas cut into sections of four and set on plastic plates were placed in the table for the duration of each question. The pizzas had unmistakably different toppings—one being a good representation of a ham and pineapple pizza, and the other providing a good representation of a pepperoni pizza.

Both researchers transcribed half of the tapes then carried out a reliability check of the transcriptions of the other half. In this way both researchers viewed all 120 video clips. Two Year 4 students' transcripts were discarded because of their incompleteness. To assist in the coding process an initial item list was drawn up, and was later refined to provide an inventory of all solutions and/or demonstrations which were explicit contrasts from other criteria. The final choice of best-fitting category for any response was by mutual agreement, arrived at through re-viewing the tape segment and/or further discussion.

Results

The questions in the assessment tasks were framed around an expectation that students evaluate, estimate or calculate a fractional part of a whole pizza in response to the request "How much pizza is left?" or "What fraction or part of a whole pizza do we each get". However, fractional responses were infrequently provided by Year 4 students in the first instance, and Year 8 students' indicated considerable unease with providing fractions that involved part pieces. Our analysis of the variety of answers raised issues concerning the use of informal knowledge, especially knowledge of whole numbers and partitioning strategies, and also examined the effect of context on students' responses. It is this effect that we wish to explore further in this paper.

Question 1 & 2: How Much of the Pizza is Left? (_ & _)

Context played a major part in many Year 4 students' deliberations. Approximately half of these students provided initial answers of "2 pieces" or "3 pieces" respectively. It would be reasonable to assume that many of these children have had both informal and formal experience of the language of a "half" and possibly "three-quarters". So why did they chose not to use this language within this context? Despite further prompting to provide a fraction approximately a third of these students either had insufficient understanding of the part/whole relationship requested or deemed that their description in terms of discrete number of pieces was sufficient. Another third of these Year 4 students provided a description of the orientation of the pieces on the dish or a description of the pizza toppings in response to the prompt "What fraction or part is left?": "There's two on the side and one up top and the bottom's missing"; "Cheese, steak, pepperoni, and tomato"; or "All the herbs".

Year 4 students were much more likely to provide an answer to the request of "how much" in terms of pieces than Year 8 students. For them, the family dinner context with its principle of sharing dominates, to the extent that the primary organisation of the *number of remaining* pieces assumes importance. If two pieces remain then these can be given to two people. It is unlikely the concept of a *half remaining* features significantly in the sharing situation. For Year 4 students in particular the interviewer's prompt is needed to provide a different perspective. In contrast, it appears that Year 8 students expect and appreciate that a mathematical response is required to the assessment task in hand and are more easily able to divorce themselves from contextual influences.

Another contextual factor evident in this question relates to the word "left". Four Year 4 students and 5 Year 8 students offered _ in response to Question 2. It is possible that they interpreted the questions as "How much of the pizza has been removed (or eaten)?"

Question 3: Altogether How Much Pizza is Left? (1_; $^5/_4$ Or $^5/_8$ Of The Two Pizzas)

In line with the propensity to offer whole number responses to Questions 1 and 2, 26 students (22%) offered a response of 5 "pieces" or "bits" of pizza; only one student offered "five fourths". The majority of students chose to express their answer as one whole and a "bit" or "a quarter".

In order to answer this question correctly students had to build a new image involving the two pizzas together. In the process of formulating totals drawn from parts of the two separate pizzas, it appears that the particulars of the pizza toppings became insignificant. In contrast to the first two questions, no student either from Year 4 or Year 8 made implicit or explicit reference to toppings. One possible explanation might be that adding fractions of pieces from two pizzas is not an everyday practice.

The dominance of responses related to "one whole" (66%) suggests that visualising a rearrangement of the pizzas (in some instances students physically shifted a piece across to make a complete pizza) was a relatively straightforward step. What was not so obvious for many students was how to refer to the "bit left over". Nine students referred to the additional pizza slice as an "extra" rather than as a fractional part:

- One whole and—one piece; a bit; a little bit more.
- One whole pizza, and that would be altogether here, and one piece left. Because if they were joined up you'd have one whole and one leftover pizza.

For some, the availability of a representational context involving concrete pieces that could be manipulated offered an opportunity to "act" and reflect on their actions:

S: A whole and a piece. A whole pizza and a half.

I: Can you tell me what fraction is left?
S: One whole pizza and one piece of pizza [rearranges pieces].
If you put this piece here so that's a whole and there's one bit left.
I: And how much is that?
S: One quarter and a whole.

Through their actions of reconstituting the pizza, and their observations and reflections of their own actions, these students engaged in understanding in action.

Question 4: [With Reference to the Remaining Five Pieces] If Four Children had a Quarter Piece of Pizza Each, Then How Much Would be Left?

While 25% of students offered a response involving number of pieces, 67% gave _ as their answer. However, given the contextual nature of the problem it would be unwise to assume that all of these students have a secure understanding of part-whole relationships—it could more simply be that "a piece" and "a quarter" are synonymous labels and that the students have effectively shared the remaining pieces by a process of removal or distribution of four pieces from the available five. One student who provided a correct fraction response was probed for an alternative meaning which, in turn, indicated that the contextual action of "sharing" was prominent:

S: One, oh, a quarter.
I: Can you think of any other way of using up the pizza? Could you give me another story?
S: It could be 4 children and a dog.
I: What a lucky dog! Bet he'd like that.

Question 5: Imagine That the Two of Us Are Going to Have an Equal Share of all of the Pizza That is Left. What Fraction or Part of a Whole Pizza do We Each get? [$^5/_8$]

The act of sharing or determining a fair share provides an everyday context in which the question of "how much" arises. Year 8 students choose to solve this problem by a number of alternative ways: physical sharing of the 5 pieces, division by 2, halving, or estimation. As with the earlier questions the majority of students used whole number partitioning strategies rather than the more formal fraction operations.

Likewise, the contextual influences related to "equal sharing" within the family dinner situation appeared to impact on many students' solution strategies. In "real" life, when sharing pizza, one likely strategy would be to distribute equal shares of the whole pieces first, before cutting the last piece. This process effectively involves partitioning the five pieces (usually into 4 + 1) and then halving each partition, resulting in a distribution of half a pizza—or two quarter-pieces—to each person and then sharing the remaining piece. Students using this approach could be interpreted as operating on the remaining pizza pieces in each step of the solution process as though they represented independent units (Kieren, 1988). Thus the solutions, "Two pieces and half a piece"; "_ a pizza and _ a quarter piece", or _ plus $^5/_8$, rather than the "tidy" mathematical form "$^5/_8$ of a pizza" would seem a logical approach—an approach that was taken by the majority (57%) of students. For many students their description of the partitioning process of the remaining piece clearly indicated that their thinking was influenced by their experiences of the contextual situation: “split this piece”; “cut this one in half”, “break it in half” “saw one bit in half; and “break this piece into two”.

Two other sharing approaches that involved responses in terms of discrete pieces rather than a single fraction of pizza included (i) partitioning of the four pieces and disregarding the “one left over piece” (10% of students), or (ii) fair sharing the pizza toppings (7% of students). It was of concern that 10% of students at Year 8 did not appreciate the need for exhaustive division. The following example illustrates one student’s reasoning as to why the 2 + 2 strategy resulted in equal shares:

> S: Half.
> I: Can you explain to me how you worked that out?
> S: I took away this piece [removes the quarter piece from the pepperoni pizza] and you have this half and I have this half [pointing to each of the half pizzas], cos if I left this pizza here [reference to the quarter piece] then I would have had more.

One possible explanation is that in real life one member of the family would in fact “receive” the extra piece as a result of being older or bigger. Alternatively, these responses may reflect students’ inability to express the halving of a pizza piece in fractional notation.

Four (7%) students interpreted the request to provide equal shares to mean an equal share of each of the pizza toppings. This pragmatic solution could reasonably be expected to be applied in a real family dinner sharing situation—or at least there would be some attempt to establish whether one person preferred one type of pizza topping. Two of these students struggled in their attempt to express their answer as fraction of the pizza:

> S: One piece of pepperoni and one piece of ham and pineapple each and half of this [points to remaining ham and pineapple piece].
> I: What would that fraction be we’d each get?
> S: 2 and a half out of 5.
> I: Can you explain to me how you worked that out?
> S: Just half of the five pieces.

The other student was more successful; he used the toppings in the first instance but then ignored the toppings in relation to his final answer:

> We would get one half of the pizza each. Oh well, we could probably divide it equally with one of each flavour and we’d put it like that [makes two half pizzas, each with one piece of ham and pineapple and one piece of pepperoni]. Then we’d cut this one in half. That’s an eighth.

Slightly fewer than half of the students provided their answer in terms of a composite or single fraction. It appears that the majority of these students intuitively applied the distributive law, partitioning the pizza pieces in much the same way as they would for whole numbers, to effectively solve the problem in two parts. The nature of the pieces (quarter pieces) meant that students were readily able to convert between the fractional name of the piece and the number of pieces—that is, the students could effectively solve this problem using whole number thinking ((4 + 1) ÷ 2) and then provide the answer as a fraction by using the fraction as a label:

> S: _ and a _ of a quarter.
> I Can you explain to me how you worked that out?
> S: Cos there’s 5 pieces of pizza, and there’s two of us, so you have 2 and I have 2 and halve this one.

Only eight (13%) Year 8 students without prompting combined _ + _ to provide an answer of $^5/_8$, one combined these to get $3^1/_{12}$, and 4 other students provided an visual estimate of $^{2}/_{3}$ or $^{3}/_{4}$. Overall, 22% of the students provided an answer in the expected single fraction format representing a "fraction of a whole pizza". If the students' explanations are an accurate reflection of their thinking it appears that the majority of students who provided an answer of $^{5}/_{8}$ solved this problem by a process of repeated halving of two separate pizzas partitions (usually reorganised as a whole pizza and a _ piece of pizza), rather than solving the problem directly as $^{5}/_{4} \div 2$ (or $^{1}/2$ of $^{5}/_{4}$). Thus despite their more sophisticated mathematical answer their solution method still involved an informal partitioning solution strategy related closely to the physical representation of the whole pizza and availability of quarter pieces.

One student provided an answer of $^{6}/_{10}$. Her explanation clearly illustrated the influence of her informal contextual knowledge that pizzas have 10 pieces [and indeed some takeaway pizzas do].

> Because a pizza is one whole, and a whole is out of 10, and we get half each, so that's a fifth and then half of one of these pieces [points to a single piece] would be 6.

Interpreting half a pizza as equivalent to 5 pieces [although she calls them fifths] and halving the extra piece to get another piece, she reasons that altogether she has 6 pieces [not of equal size]. She then uses a part/whole interpretation of a fraction to justify that each person would receive $^{6}/_{10}$ of a pizza.

Three other students physically reorganised the pizza pieces and visually estimated the proportion of pizza share as $^{2}/_{3}$. In the following explanation an answer of "about two thirds" is possibly indicative of the accuracy required and process used in a "real" situation:

> S: $^{2}/_{3}$ of a whole pizza.
> I: Can you explain to me how you worked that out?
> S: Well, if each eat half of what was left, that's 5 pieces, so if we half that we get 2_ pieces and if you have 2_ pieces here, _ a piece would add to this [overlaps a half a piece onto the half pizza] and would come round to about there, which would be about two-thirds.

Those students who did attempt some sort of algorithmic calculation invariably encountered difficulties. Many of these students accepted a calculation even when that calculation was at odds with their physical demonstration. The acceptance of nonsensical answers suggested that for some students the need to justify or provide explanations is not a common practice. Rather, these students relied on the belief that the authorised mathematical way is more correct than their own intuitive reasoning related to the contextual nature of the problem.

Implications and Conclusions

Rational number understanding and the development of the complex ideas which are fundamental to that understanding emerge at a number of different levels and in different ways. In this study students at the two different Year groups demonstrated that this development is very much a function of time and associated educational experiences. There are conceptual difficulties associated with shifting from informal knowledge to the concept of part/whole distributions and students need to recognise that something quite different from mere additive reasoning is demanded. The familiar context of the family pizza dinner does not necessarily promote that recognition. The differences between rational number thinking and family dinner distributions are subtle, yet recognising those differences and moving beyond them allows the development of mathematical fraction thinking to emerge. Notwithstanding, meaningful apparatus and manipulatives, such as the circular pizza measurement model, do have a part to play in the development of rational number understanding, but this development requires explicit pedagogical discussion linking the concept with the structure. Most circular representations used in the classroom model a continuous region. However, because the model in this research was sectioned into movable familiar pizza pieces, many of the students interpreted the questions as "how many?" rather than "how much?" Only when the connections between concept and structure are firm and stable will students be able to evoke the generalisability of the part/whole distributions.

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