

Copying on a Graphics Calculator and Implications for Mathematical Understanding

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The subject of inquiry in this paper is copying on a graphics calculator by students in a Year 12 Calculus class. They copied interim numerical results and algebraic expressions as part of calculation. The practice called for knowledge of order of operations, was a means for linking representations and generalisation, and was a facet of discernment of function relationships. Implications for teaching are the act of copying could assist the development of understanding in each of these domains.

In this paper, I report on the practice of copying on a graphics calculator: in other words, transporting numerical values and algebraic expressions into calculation other than by keying them directly. I identify four methods of copying used by students in a Year 12 Calculus class, explore why copying was valued in the class, and explore the implications of copying for mathematical understanding. The paper is part of a systematic inquiry into the practices and understanding evidenced by students in the Year 12 class, in relation to graphics calculator use. At the time of the fieldwork, the students had owned and used graphics calculators for nearly two years. The brand was the Hewlett Packard HP38G, which has limited symbolic capabilities. The teacher was experienced in teaching with the calculators and included use of them in instruction. I attended the class for 21 lessons, observing whole-class work and acting as an assistant teacher during individual work.

The assumptions of the *emergent (constructivist) perspective* of Cobb, Boufi, McClain and Whitenack (1997) underpin the analysis. Cobb et al. propose that learning evolves over time and involves social and mental processes, with neither taking precedence. So, learning comprises (social) enculturation into mathematical ways of being and (mental) advancement in the ways that mathematics is understood. However, learning in the social and mental domains is not distinct: activity in one domain reflexively informs activity in the other. The enculturation that I address is copying in calculation and the mathematical advancement that I discuss mainly concerns function relationships.

Further, I assume operational and structural views of mathematical understanding (Sfard, 1991); that is, understanding in terms of processes and concepts (Gray & Tall, 1994). Operational understanding is salient when mathematical objects are perceived in terms of operations on other objects. Structural understanding exists when mathematical objects are perceived as entities with properties. For instance, $f(x) = x^2$ would be understood operationally as entailing a squaring process, and recognising from the x^2 that $f'(x)$ has a single turning point would indicate structural understanding.

The significance of this paper for mathematics education is the identification of copying on the graphics calculator and identification of potential benefits of copying for mathematical understanding.

The Literature

Schwarz and Dreyfus (1995) identify that use of technology in mathematics requires that students recognise mathematical objects in new ways and that three categories of skill assist the recognition. The first category comprises skills for dealing with partial data, viz. incomplete graphs and tables of values with few values. Second there are skills that enable students to link properties recognised in one representation with properties recognised in another. Third there are skills for carrying out transformations in one representation. In essence, copying on a calculator belongs to the third skill category. It entails copying numerical results into new calculations, and copying algebraic expressions in order to compose new expressions. So, it involves transportation of values and expressions, prior to their transformation, in numeric and algebraic settings.

Heck's (1995) analysis of the meaning of variable in relation to graphics calculator use is highly relevant to the practice of copying. He points out calculator use requires a wider conception of variable than in conventional mathematics. For example, memories on the graphics calculator are known by the letters of the alphabet, A, B etc., and the unique values stored in the memories are copied into calculation when the letters are used. In comparison, often letters (or variables) are held to stand for sets of values (i.e., generalised number) in conventional mathematics. Students need to appreciate the different meanings. Furthermore, Graham and Thomas (2000) describe an activity for developing understanding of the unique value and generalised number conceptions on graphics calculators. Students stored values in A and B memories and evaluated expressions, for example $A + B$ and A/B , and repeated the procedure.

Heck (1995) also notes that some variables on the calculator have special meaning, and the notation is not a single letter, for example, ANS for last answer that was calculated. As well, functions, matrices, and lists of numbers can be variables and, like ANS etc., can be copied by using their notation and then can be transformed. For instance, adding 10 to a list variable increases every number in the list by 10. Transformation, in Sfard's (1991) terms, means the variables are operated on. They are treated as mathematical objects. Furthermore, according to Cobb et al. (1997) and Sfard (1991), treating something as an object can assist understanding it as an object. The treatment (external to the person) and understanding evolve together.

Doerr and Zangor (2000) report an introduction to continuous rate of change that utilised function variables. Students constructed the functions $y_2(x)$ for functions $y_1(x)$ using the formula $y_2(x) = [y_1(x + \Delta x) - y_1(x)] / \Delta x$ in their graphics calculators. The variable y_1 was the means for copying the function into the y_2 calculation. Students set Δx to 0.1 or 0.01 and they inferred rate of change (i.e., derivative) properties from the table of values and graph of $y_2(x)$. Yerushalmy and Schwartz (1999) recommend a similar approach and Lagrange (1999) discusses applications that utilise function variables on a CAS (computer algebra system).

In summary, I take "copying on a calculator" to be a skill that involves the transportation of numerical values and algebraic expressions so that they are ready for transformation. It can be achieved by naming calculator memories and function, matrix

and list variables. The practice calls for a wider conception of variable and could support the development of understanding of function and other mathematical relationships.

Research Method

I attended 21 lessons in the Year 12 class. I observed whole-class work and was an assistant teacher during individual problem-solving. The teaching role gave me opportunities to query students about their calculator use and I made brief field notes. I set up four audio-recorders to capture the conversations of the 13 students in the class, who generally sat in four groups. I transcribed at least some of the audio-recordings immediately after the lessons and followed up issues with the students and teacher during or after subsequent lessons. I set up a video-recorder to record continuously, with the display from the overhead projector in the field of view. This data source allowed me to replicate graphics calculator use during instruction because the teacher frequently connected his own or students' calculators to a projection panel.

I kept a research journal and recorded aspects of calculator use that caught my attention. I developed a list of types of use, including copying, and at the end of the fieldwork systematically searched the data for each type. At the reporting stage, I proposed in writing to the teacher the scope of the analyses for publication and we discussed and agreed to the proposal. I ask him to scrutinise each paper, and seek critical feedback from at least one colleague with knowledge of graphics calculator use.

Analysis

Copying Stored Results and Expressions

The Year 12 students used four methods for copying. The first was to proceed with calculation after an interim result had been obtained by keying the next operation. See Figure 1 which shows a student's calculation for solving $3 = 4e^{-k \times 100}$ for k . Keying '/' after the logarithm result activated 'ANS', which copied down the $-.287..$

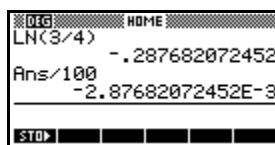


Figure 1. Solving $3 = 4e^{-k \times 100}$ for k .

The second method involved highlighting an interim result, and using the COPY key. The copied result was used in its entirety or unwanted parts were deleted. The method is illustrated in Figure 2 in relation to projectile motion given by $\mathbf{r} = 40t\mathbf{i} + (40\sqrt{gt} + 5t^2)\mathbf{j}$. A student was calculating the horizontal distance from the point of projection to point of impact on the ground. She found the roots of the \mathbf{j} component using the POLYROOT function, copied the result, isolated the required root, 13.856.., and multiplied by 40 to yield the horizontal distance. She obtained POLYROOT from a menu of functions.

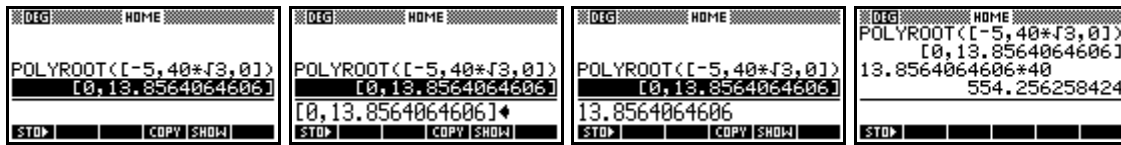


Figure 2. Copying by highlighting, then, deleting the unwanted root.

Students also copied expressions by highlighting, typically when an expression similar to one that they had already entered was needed in calculation. Unwanted parts of the copied expression were deleted and new parts keyed. Figure 3 shows the method for the evaluation of $\int_0^3 (-5\pi/2)\cos(\pi t/2)\mathbf{i} + (-5\pi/2)\sin(\pi t/2)\mathbf{j} dt$. A student suggested it when the teacher claimed the integral had to be done by hand. The HP38G calculator that students were using didn't calculate integrals for vector expressions.



Figure 3. Copying by highlighting, then, changing cosine to sine.

The third method was to name the memory in which a result was stored. For example, a student used 'A' in calculation, and this called up the value of 'A' that she had just calculated in the numeric solve facility (Figure 4).

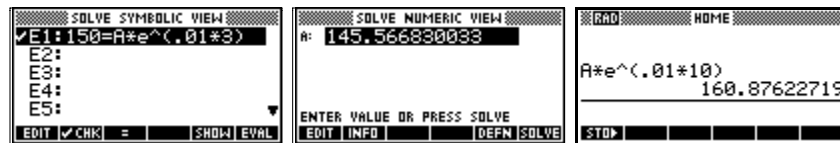


Figure 4. Copying by naming memory 'A'.

Students also used ROOT as a limit of integration, after having just obtained the root or of the function they were integrating. The value was automatically stored in the ROOT memory. ISECT was used the same way, for the x co-ordinate of a point of intersection.

The fourth method was to copy functions by referring to function variables. For example, a student had entered $f(x) = \cos x / (2 + \sin x)$ in the graphing facility of her calculator to determine where the "area bounded by the curve and the x axis" was above the x axis. Then, she calculated the area in the calculation facility (HOME), using the function variable notation F1(X) (see Figure 5).

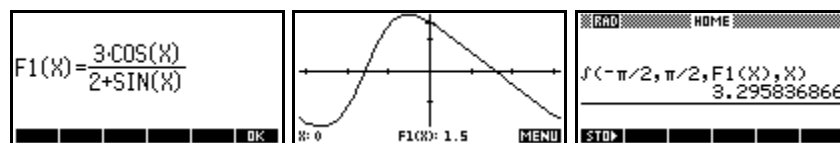


Figure 5. Copying using function notation.

In other instances, students used the function variable notation in HOME with a value assigned to the variable, which effected evaluation. For example, when the teacher

set the task of calculating speed on $\mathbf{r}(t) = 10\sin t \mathbf{i} + 5\sin 2t \mathbf{j}$ at various times, a student entered the components of velocity into the parametric graphing facility of her calculator and calculated the speed using the absolute function $\text{ABS}((X2(\theta/2), Y2(\theta/2)))$, which produced the speed at $t = \pi/2$ (see Figure 6).

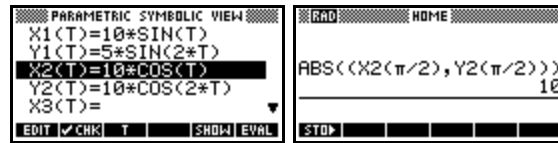


Figure 6. The screen displays for the graph of $\mathbf{r}(t)$ and the speed calculation.

Reasons for Copying in the Year 12 Class

Copying can be efficient: it reduces the keystrokes, transcribing errors are avoided and all the decimal places are carried through intact (e.g., see Figure 1). The teacher promoted copying on the basis of efficiency, particularly when students' calculations were projected on the whiteboard and were visible for all to see. He made comments like "that's the quickest way to do it" and pointed to errors when students used rounded interim results instead of copying full results into calculation. He was also critical of copying for repeated calculation and asked for more efficient ways. These were using generalised equations in the function graphing facility or numeric solve facility. Further, twice while I was observing the class students raised the problem that the fourth copying method can be slow in producing results.

In fact, many students moved to store a generalised equation for exponential calculations, and $A = Ie^{kt}$ was voiced in class discussion when the teacher asked for a quick method for the calculations. After entering three of the parameters/variables, the fourth was evaluated through use of the SOLVE command (see Figure 7).

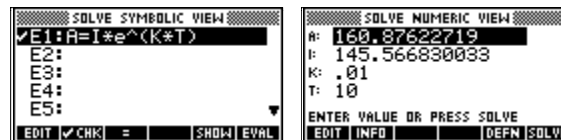


Figure 7. The screen displays for the exponential equation.

Also in class discussion, the teacher asked for a quick method for the repeated speed calculations (see Figure 6). A student said to use T instead of $\pi/2$ and store values for T. On inquiry, she said to store T values by placing the cursor on the $\mathbf{r}(t)$ graph at the points for which speed was to be calculated. The class had previously plotted the $\mathbf{r}(t)$ graph (see Figure 8). Another student suggested to put the speed expression in the function graphing facility. Functions to calculate the argument (angle from the horizontal axis) for the velocity vector (X2(T), Y2(T)), in radians and degrees, were added and students saved the set-up (see Figure 8). They used it to obtain speed and argument values from table of values and the speed graph was used in prediction (see Forster, 2003).

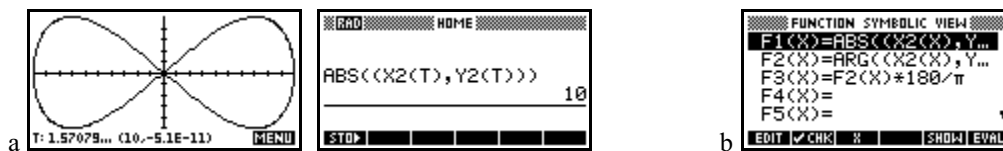


Figure 8. (a) Speed calculation at $T=1.57079\dots (\pi/2)$ and (b) speed and argument generalisations.

The class also had other applets stored from before I attended the class. One calculated and graphed the first and second derivatives of functions that students entered (see Figure 9). The set-up saved students keying the derivatives, and the lines $F3(X)=F2(X)$ and $F5(X)=F4(X)$ allowed the evaluation of derivatives while the derivative syntax in $F2(X)$ and $F4(X)$ was left for future calculations. The class had originally used the set-up for exploring rectilinear-motion relationships. Another applet produced consecutive approximations for the Newton Rhapsom method. It used function notation, was set up for repeated use and had been designed in class to facilitate calculation.

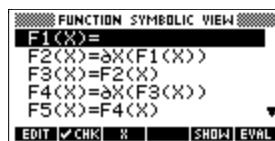


Figure 9. The screen displays with expressions for first and second derivatives.

Hence, the statements in class in relation to copying, and the nature of applications that utilised the practice (e.g., Figures 7-9), indicated it was valued for quickness and accuracy in calculation, and was rejected when it did not meet these criteria. In addition, copying facilitated and might have influenced students to calculate in steps (see Figure 1) and had supported investigation.

Implications for Mathematical Understanding

Order of operations. A step approach (see Figure 1) starts with decomposing objects of calculation and relies on knowledge (viz. operational understanding) of order of operations. Applications on the calculator could also be used to develop the understanding in middle school years. A simple instance would be asking students to evaluate $\frac{8-1}{7-3}$ two different ways on the calculator and to check the results against each other. Alerting students to copying would assist the endeavor. Then, repeatedly calculating in steps on the calculator, as did the Year 12 students, would consolidate understanding of order of operations. Analysing the cause of erroneous results would assist the consolidation.

Parameters and variables. As Heck (2001) observes, graphics calculator use calls for an expanded understanding of “variable”. In the examples of calculation given above, A and other letters stood for parameters, that is, given conditions (see Figure 3 where A was initial population); or they stood for variables (see Figure 7, where A was the amount or population over time). Bringing students’ attention to using symbols interchangeably for parameters and variables, in the context of copying, could help them realise that the symbol itself does not indicate the parameter/variable status. The

distinction between parameters and variables is important in advanced equation-solving and is an area where students fall down (Drivjers & van Herwaarden, 2000).

Function properties. Calculating with function variables (e.g., $F1(X)$, $X2(X)$, Figures 5, 8 9) meant the functions as wholes were operated on. Sometimes the result was another (new) object (viz, absolute value, argument, a derivative), and properties of the old and new objects could be discerned. An example of discernment was a student used the argument function $F2(X)$ in Figure 8 to determine time for minimum height on an $\mathbf{r}(t)$ trajectory. She entered the velocity components for the motion in the parametric facility as $X2$, $Y2$, then, graphed the argument function and located the root. Thus, she used the property that velocity is horizontal at the point of minimum height.

The table of values and the graph for the set-up in Figure 9 were both used in the investigation of rectilinear motion properties. The table, with values for the function and first and second derivatives, reveals, for instance, that velocity can be negative in the presence of positive acceleration. The set-up in Figure 9 can be seen to be a variation to the derivative formulations described by Doerr and Zangor (2000) and Yerushalmy and Schwartz (1999). Importantly, the form of expressions with function variables (e.g., $\ddot{X}(F1(X))$) might help students appreciate that many graphical properties apply to functions and their derivatives per se, rather than to specific functions. In fact, it is known that symbolic form is highly influential to the meanings that students' develop (e.g., Groisman, Shapiro & Willinsky, 1991).

Linking different representations. The function variables were a means for linking different representations in the calculator. In the case of integration, the need to produce a graph (as in Figure 5) was motivation for entering the function expression as $F1(X)$, and $F1(X)$ appeared on the graphing screen (see the bottom of the graphing screen, Figure 5). Then, using $F1(X)$ in the integral meant the integral was directly linked in the calculator to the expression and graph. Simple applications like this prepared students for more sophisticated formulations, where one result was the progression illustrated in Figures 6 and 8, which finished with the speed-time graph. The ability to link different representations assists problem-solving and the skills required fall into Schwarz and Dreyfus's (1995) second category: skills for translations between representations. Hence, copying can be classified as being in Schwarz and Dreyfus's second skill category as well as the third.

Generalisation. The use of function variables also mediated algebraic generalisation. In the instance of the speed calculation (see Figure 6), the first step was calculation for a specific time $\pi/2$. Then, a student suggested to use T , where T stood for the time corresponding to the position of the cursor on the $\mathbf{r}(t)$ graph. The teacher asked for a quicker way and the generalisation was produced. Moving from numerical values, to "variable" as a place holder for a specific number, to the generalisation indicated advancement to (structural) understanding and allowed flexibility in problem-solving.

Albeit, the generalisation for speed $\sqrt{100\cos^2 t + 100\cos^2 2t}$ for $\mathbf{r}(t) = 10\sin t \mathbf{i} + 5\sin 2t \mathbf{j}$ could have been used, but this would not have entailed the

direct link to $X2(T)$, $Y2(T)$ in the parametric graphing facility. The direct link allowed students to apply the generalisation to any $\mathbf{r}(t)$.

Concluding Discussion

Copying is a skill that effects the transportation of numeric values and function expressions into calculation, ready for transformation. As a result, copying can optimise efficiency in calculation. In fact, during the time that I was in the Year 12 class, copying was valued for efficiency and the teacher promoted it through explaining and critiquing students' calculations that were projected on the whiteboard. Thus, the practice was publicly available for students who might not have had it, with all its variations, in place.

In the act of copying, students brought to bear order of operations, used letter symbols in the place of parameter and variable values, linked representations and formulated generalisations. Therefore, in the view that action can lead to understanding, copying warrants consideration as a means for developing understanding in these domains. Further, copying was a context for the inference of function properties. In conclusion, an important implication for teaching is that activities which utilise copying and explicitly address the development of understanding may be beneficial. They warrant consideration and future research.

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