Probing Whole Number Dominance With Fractions

Max Stephens
The University of Melbourne
<m.stephens@unimelb.edu.au>

Catherine Pearn
Catholic Education Office, Melbourne
<cpearn@ceo.melb.catholic.edu.au>

Children's whole number schemes can interfere with their efforts to learn fractions. To what extent do these schemes persist for secondary school students? This paper reports on the development and piloting of an interview designed to identify and probe inappropriate whole number strategies for working with fractions among secondary students. The interview showed that these strategies are still prevalent among Year 8 students. Among others who use appropriate multiplicative strategies the interview showed that some of these are still not confident in challenging instances of inappropriate whole number thinking.

Behr, Wachsmuth, Post, and Lesh, (1984) define "whole number dominance" as thinking which involves "making separate comparisons of numerators and denominators using the ordering of whole numbers" (p. 332). Hart (1981) notes that "a fraction of course involves two whole numbers which have to be dealt with as if they were irrevocably linked" (p.69). The ratio between numerator and denominator is the "irrevocable" link. Other researchers have noted how children's whole number schemes can interfere with their efforts to learn fractions (Hunting, 1986; Streefland, 1984; Bezuk, 1988). Kieren (1980; 1988) suggested that difficulties experienced by children solving rational number tasks arise because rational number ideas are sophisticated and different from natural number ideas and that children have to develop the appropriate images, actions and language to precede the formal work with fractions, decimals and rational algebraic forms. Saenz-Ludlow (1994) maintained that students need to conceptualise fractions as quantities before being introduced to standard fractional symbolic computational algorithms. Streefland (1984) discussed the importance of students constructing their own understanding of fractions by constructing the procedures of the operations, rules and language of fractions.

An Australian research project was designed to investigate the extent to which children's thinking processes might be associated with qualitative differences in their whole number knowledge when solving rational number tasks (Hunting, Davis, & Pearn, 1996). This research highlighted the vast difference in the children's mathematical knowledge and the type of whole number strategies they used when solving mathematical tasks. The most successful students solving whole number tasks, were more successful and used superior strategies when solving rational number tasks. Students who relied on rules and procedures when solving whole number tasks were less successful with rational number tasks. They experienced some success with partitioning and ratio tasks but little or no success with fraction tasks set in various contexts.

This paper focuses on reliance on strategies used by secondary school students during the piloting of a Probing Fraction Interview (Pearn & Stephens, 2003).

Development of an Assessment Package for Students From Years 5 -8

Two Fraction Screening Tests (Pearn & Stephens, 2002a, 2002b) were designed as broad assessment tools for teachers to use with a whole class group in order to identify areas of strengths and weaknesses. The tasks included contexts such as discrete items, lengths, fraction walls, and number lines. Fraction Screening Test A (Pearn & Stephens, 2002a) was designed mainly for students in Years 5 and 6 and for weaker students in Years 7 and 8. Fraction Screening Test B (Pearn & Stephens, 2002b) was intended for students in Years 7 and 8 or higher achievers in Years 5 and 6. There were 20 items altogether in each Screening Test with 11 common items, and extension items in Test B only.

Results from Screening Test B (see Figure 1) showed that Year 7 and 8 students were successful with tasks presented in conventional contexts such as shading three-fifths of an unmarked rectangle; and with the fraction one-third, for example, finding the whole given a third using discrete objects.

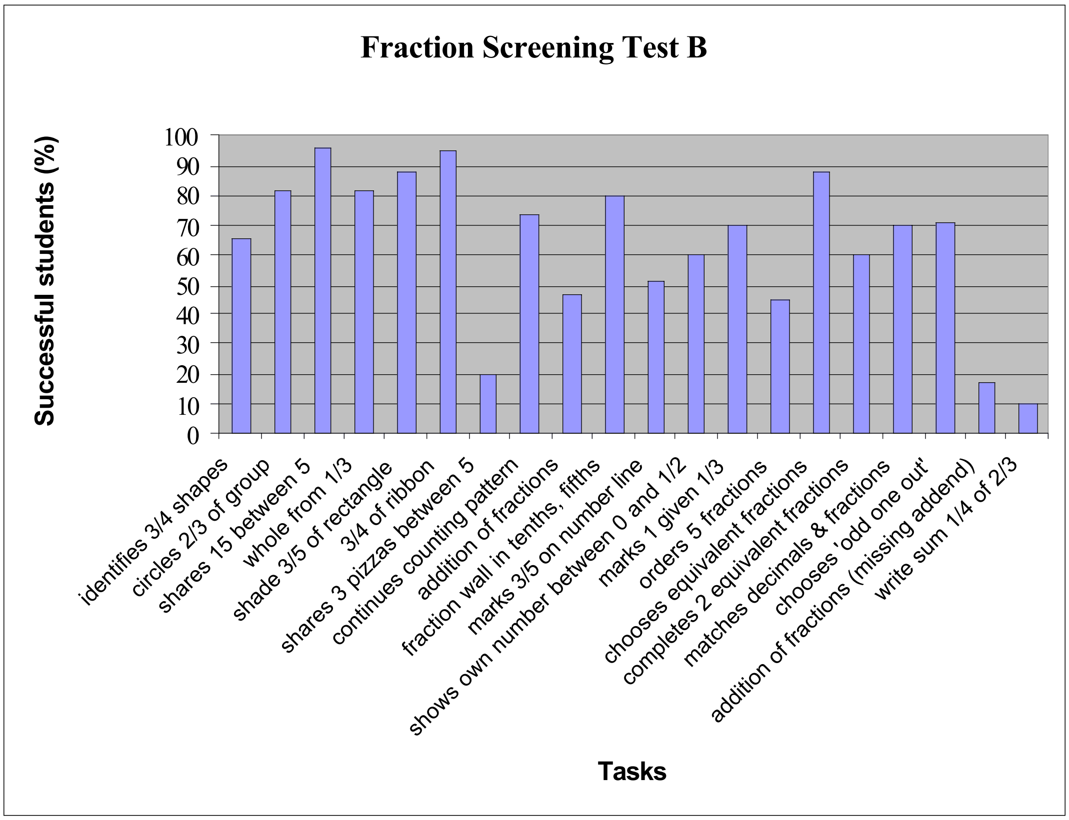


Figure 1. Results from Screening Test B.

They were less successful with tasks that involved fractions as numbers, for example "Put a cross (x) where you think the number $\frac{3}{5}$ would be on the number line". Many students interpreted this question as requiring them to find three-fifths of the entire line ignoring the numbers zero, one and two marked on the line.

Our Research Conjecture

Given the relatively poor performance of Year 7 and 8 students on particular items of Screening Test B, we looked for evidence of some thinking strategies that had lead to these poor results. While screening tests show patterns of strengths and weaknesses, they generally fail to disclose the kinds of thinking used by students. A one-to-one Fraction

Interview (Pearn & Stephens, 2002) had been developed to ascertain students' knowledge of rational numbers. The tasks included contexts such as discrete items, lengths, fraction walls, and number lines.

We conjectured that inappropriate whole number thinking strategies were being commonly applied to fraction problems. For example during a video-taped interview (Pearn & Stephens, 2002), Robert, a Year 7 student, who otherwise showed sound conceptual and procedural understanding of the fraction tasks, gave an unexpected explanation. When asked why he had decided that $\frac{2}{3}$ was larger than $\frac{3}{5}$, Robert said, "From 2 to 3 is one and from 3 to 5 is two so $\frac{2}{3}$ is bigger than $\frac{3}{5}$".

Robert's explanation shows whole number dominance as Behr et al. (1984) define it. In the instances of "whole number dominance", referred to by Behr et al., students typically calculated the difference or "gap" between numerator and denominator to compare fractions. In this study, we show that whole number dominance includes other strategies where students deal with numerators and denominators, ignoring the ratio connecting numerator and denominator. By contrast, we define *multiplicative thinking* as those strategies which preserve the fundamental ratio between numerator and denominator.

A different interview was needed to reveal evidence of inappropriate whole number thinking among a group of secondary students who had been studying fractions for at least four years, and who had been meeting fractional ideas for an even longer. The rest of this paper deals with the development and field trial of this interview with a group of students in Year 8. The following questions guided the development of the interview protocol:

- Was this kind of thinking just a phenomenon of a particular student or do some other capable students still use these strategies?
- How can we probe more deeply this tendency to fall back on inappropriate whole number strategies among quite capable students?
- How well are students able to identify difficulties that arise if inappropriate whole number strategies are used?

Developing a Probing Interview

A Probing Fraction Interview was designed to address these questions among students in Year 8. This interview has three parts:

Part A: baseline questions for Year 8 students

Part B: questions intended to disclose inappropriate whole number thinking

Part C: scenarios inviting students to critique inappropriate whole number thinking

Parts A and B were given to all students. Only if students successfully completed these were they given Part C. In Part A there are nine baseline tasks that include the recognition and completion of equivalent fractions, ordering fractions, fractions as numbers and matching fractions with decimals. These basic questions were designed to give a broad picture of each student's knowledge of fractions. An example of a Part A task is as follows:

Point to the cards with pairs of fractions: $\frac{2}{5}$ $\frac{2}{3}$ $\frac{1}{4}$ $\frac{2}{8}$ $\frac{1}{3}$ $\frac{2}{3}$
Interviewer: *Which of these pairs of fractions are equivalent?*

Wait for student to respond. *How did you decide?*

In Part B, four tasks were designed to probe inappropriate whole number thinking. In particular these tasks focus on choosing the larger fraction, completing equivalent fractions, and deciding on the input required for a given output from a "fraction machine". For example in one task, the interviewer pointed to two cards: $\frac{3}{5}$ $\frac{2}{3}$ and asked: "Which is larger: three-fifths or two-thirds?" After the student responded, the interviewer asked: "How did you decide?"

Part C of the Probing Fraction Interview consisted of four scenarios that used actual student responses embodying inappropriate whole number thinking. The scenarios asked students to identify the difficulty experienced by another student and where possible to give an example which would clarify the thinking of the student in the scenario. One such scenario presented Robert's thinking as discussed earlier. In the scenario students were asked, "What would you say to Robert about the way he got his answer? Can you think of an example that you could use to explain why Robert's method does not always work?"

Trialling the Probing Interviews

To test the instrument, twelve Year 8 boys from a metropolitan Catholic secondary school were interviewed by the authors. These students were identified by their teachers as being either "average or above" in mathematics. While the Probing Fraction Interview was conducted, one researcher interviewed the student and another recorded what was said.

Results

Three distinct groups of students were identified by this interview. The first group can be described as *proficient* multiplicative thinkers. These students showed sound conceptual and procedural thinking about fractions. A second group can be described as *residual.* They showed some inappropriate whole number thinking even though they generally used correct multiplicative strategies. The third group can be best described as *default* whole number thinkers. Students in this third group occasionally showed some correct multiplicative thinking, but tended to rely on inappropriate whole number thinking. In this section responses from six students, two from each group, will be presented.

Several students who can be described as *default* whole number thinkers showed misconceptions on some Part A tasks which were generally done competently by *residual* and *proficient* thinkers. For example, Sean and Matthew used inappropriate whole number thinking when dealing with equivalent fractions and ordering fractions.

Sean (*default*): $\frac{1}{4}$ and $\frac{2}{8}$ are equivalent fractions.

They can go into each other: 1 goes into 2 and 4 goes into 8

Interviewer: "Are there any other equivalent fractions?"

Sean: $\frac{2}{3}$ and $\frac{1}{3}$ are equivalent. Interviewer: "Why is that?"

Sean: 1 goes into 2 and 3 goes into 3. The other pair are not equivalent

Sean's difficulty is to regard the numerators as forming one pattern and the denominators another, ignoring the ratio numerator:denominator (see Hart, 1981, p.69). On the other hand, Matthew uses the "gap" between numerator and denominator to compare fractions.

Task: $\frac{2}{3}$ $\frac{3}{4}$ $\frac{5}{8}$ Place these cards in order from smallest to largest.

Matthew (Default): 2 away from 3 is less, so $\frac{2}{3}$ is the larger, 8 is bigger than 5 … more to go into, so $\frac{5}{8}$ is smaller, and $\frac{3}{4}$ is in the middle

In Part B students who can be described as *default* whole number thinkers gave consistently incorrect responses to these tasks which were designed to reveal inappropriate whole number thinking. Students who can be described as *residual* whole number thinkers generally completed this section using appropriate multiplicative approaches such as Lowest Common Denominator for comparing fractions or completing a "chain" of equivalent fractions such as

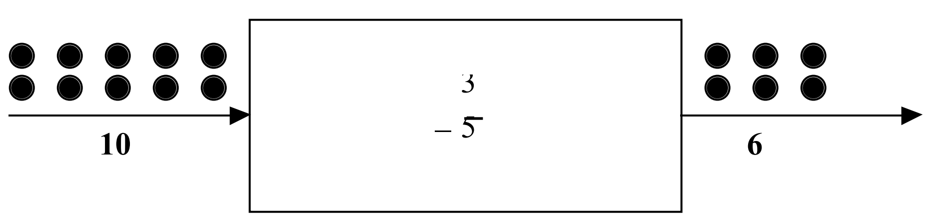
$$\frac{1}{4} = \frac{2}{8} = \frac{3}{\square} = \frac{\square}{20}$$

In the example below, Sean uses the product of each numerator and denominator to compare fractions whereas Matthew uses the difference (the "gap") between numerator and denominator. Sean and Matthew correctly recognise that a comparison of fractions involves both numerators and denominators. Their use of inappropriate strategies is the problem.

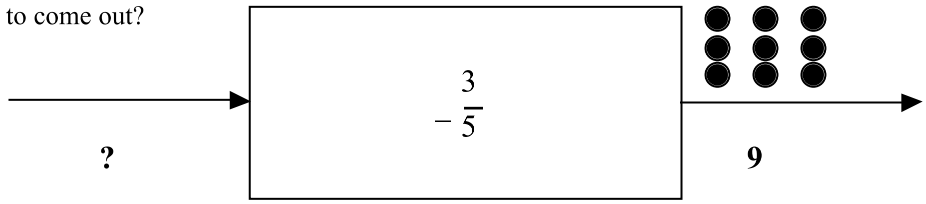
Which is greater $\frac{2}{3}$ or $\frac{3}{5}$?	
Sean (default):"Maybe 3 _ 5 = 15 and 2 _ 3 = 6 so $\frac{3}{5}$ is larger."	Matthew (default): $\frac{2}{3}$ is larger … smaller numbers … less to go into … 2 to 3 is 1, 3 to 5 is 2"

Both *default* whole number thinkers and *residual* whole number thinkers found difficulty with the "number machine" problem in which they were asked to reverse a fraction operation equivalent to multiplying by three-fifths.

Here is a three-fifths number machine.



When I put 10 objects in the machine 6 objects come out. How many objects would I need to put in for 9 objects to come out?



Stephen (*proficient*) and Elias (*proficient*) were able to successfully complete this task. Stephen divided the output (9) by 3 to show that $\frac{1}{5}$ is equivalent to three objects and then used $\frac{5}{5}$ to argue that the input was 15 objects. Adrian (*residual*) said: "10 objects in 4 less out, so 13 in to get 9 out." Sean (*default*) said: "10 minus 6 is 4, so 9 + 4 = 13."

In Part C, differences emerged between *proficient* multiplicative thinkers and those who can described as *residual* whole number thinkers. *Proficient* multiplicative thinkers were able to challenge inappropriate whole number thinking presented in the scenarios, and were able to provide an example which showed that method used in the scenario does not always work.

Robert said that $\frac{2}{3}$ is larger than $\frac{3}{5}$ because:

from 2 to 3 is one (point to $\frac{2}{3}$) from 3 to 5 is two (point to $\frac{3}{5}$)so two-thirds is larger than three-fifths.

Is his answer correct? What would you say to Robert about the way he got his answer?
Can you think of an example that you could use to explain why Robert's method does not always work?

In the following example, Adrian (*residual*) both endorses Robert's method and also uses the size of the denominator to determine which fraction is smaller.

Adrian (*residual*): "Robert's answer $\frac{2}{3}$ is correct. The smaller the denominator the bigger the fraction. You can do it that way if you want to, but it's simpler to look at the denominator."

Paul (*residual)* like Robert, uses the "gap" between numerator and denominator to compare two fractions. He also uses whole number language to refer to fractional parts.

Paul (*residual*): "Robert's answer $\frac{2}{3}$ is correct. Yes. For $\frac{2}{3}$ you only need 1 (*sic*) to make it a whole, for $\frac{3}{5}$ you need 2 (*sic*) more to make it a whole."

Unlike Robert, Jennifer says that $\frac{3}{5}$ must be larger than $\frac{2}{3}$ because three is larger than two (point to the numerators) and 5 is larger than 3 (point to the denominators).
Is her answer correct? What would you say to Jennifer about the way she got her answer?
Can you think of an example that you could use to explain why Jennifer's method does not work?

Stephen (*proficient*): Jennifer's rule can be shown to be incorrect. If you took $\frac{1}{2} = \frac{3}{6}$ then by increasing the numerator to 4 and the denominator to 20, this leads to a smaller fraction. ($\frac{4}{20} = \frac{1}{5}$).

Stephen (*proficient*) gives a counter example to show that Jennifer's thinking would lead to one-fifth being greater than one-half. Paul *(residual*), while disagreeing with Jennifer's method, still accepts Robert's method as a reliable method for comparing fractions. It should also be noted that Paul uses whole number language in the example below to refer to fractional parts.

Paul (*residual*): "Jennifer's rule is not correct. You either need Lowest Common Denominator or Robert's method. If you add one more to $\frac{3}{5}$ you get $\frac{4}{5}$."

Findings

The Probing Fraction Interview exposed a range of inappropriate whole number strategies for comparing fractions. All four questions in Part B identified clearly those whose thinking can best be described as *default* whole number thinking. Those whose thinking has been described as *residual* whole number thinking did well on this section, except for the last question which asked them to reverse the operation of a three-fifths number machine. It was in Part C that *residual* whole number thinkers experienced greatest difficulty. Unlike *proficient* thinkers, they were often unable to mount a clear and convincing challenge to the scenarios presented. They were, however, often able to draw upon alternative strategies to complete the task correctly. A defining feature of *default* thinkers appears to be a lack of checking.

Findings from the Probing Fraction Interview can be summarised as follows. *Proficient* multiplicative thinkers understand that fractions can be represented in a range of equivalent forms and use multiplicative thinking to relate fractions and whole numbers. They know how to represent fractional numbers on the number line and are able to compare and order fractions using common denominators. *Proficient* multiplicative thinkers are able to match fractions and equivalent decimals. They correctly perform algorithms involving fractions. They can reverse fractional operations by linking a group of objects to a fraction of a whole. They are able to challenge inappropriate whole number thinking and can provide convincing counter examples.

Residual whole number thinking refers to students who are generally multiplicative thinkers. They understand that fractions can be represented in a range of equivalent forms, but tend to revert to inappropriate whole number thinking when faced with new or unfamiliar fractional problems. They use multiplicative thinking to relate fractions and whole numbers and can represent fractional numbers on the number line, but sometimes confuse a fraction number with a fractional part of the number line when the line is longer than 1 unit. *Residual* whole number thinkers may use inappropriate whole number thinking to compare and order fractions using differences ("gaps") between numerators and denominators. They are able to match fractions and equivalent decimals but tend to look at the differences between whole and part when asked to reverse fractional operations.

Residual whole number thinkers are less confident in challenging inappropriate strategies. However, they are able to call upon alternative methods to check their reasoning.

Default whole number thinking refers to students who treat numerators and denominators in ways that ignore the ratio between numerators and denominators. As a result they use inappropriate strategies involving numerators and denominators. They may not understand that fractions can be represented in a range of equivalent forms. They tend to use guesswork or estimation to relate fractions and whole numbers, and confuse fractional numbers with a fractional part of the number line. *Default* whole number thinkers use inappropriate whole number thinking to compare and order fractions, such as focussing on differences ("gaps") between the numbers in the numerator and denominator, or otherwise working with numerators and denominators in ways which ignore the ratio between numerator and denominator. They are not confident in matching fractions and equivalent decimals and tend to look at the differences between whole and part when asked to reverse fractional operations. They are usually unable to call upon alternative strategies to check their reasoning.

Conclusions

The Probing Fraction Interview showed that inappropriate whole number strategies involving numerators and denominators are prevalent among students in Year 8. These strategies can all be considered instances of whole number dominance in the sense that they ignore the fundamental ratio between numerator and denominator. Three broad categories of performance were identified by the trial of the interview. The trial does not allow us to identify the proportions of students who fall into each category. Our conjecture, however, is that too many students appear to fall into either *residual* or *default* categories. The strategies they use are unlikely to be disclosed by regular pencil and paper testing. In a busy Year 8 program there may be few further opportunities to remedy the misconceptions displayed by *default* whole number thinkers.

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