A Model of Early Number Development

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Children sometimes struggle to learn that counting will tell them "how many". The literature in relation to children's early number knowledge is reviewed in order to identify components in children's understanding of number. A model is then proposed of how these different number components come together for children as they develop an understanding of numbers as a representation of quantity.

Teachers recognize that counting is critical to children's understanding of numbers and often assume that it is through counting that they come to understand that numbers can be used to represent quantities. Counting, therefore, is typically the focus of mathematics lessons when children first come to school. Learning to use numbers to say, *with meaning*, how many there are in a collection, however, can be a major hurdle for many children. The study from which this paper is drawn (Treacy, 2001) sought to identify what number knowledge supports children's developing understanding of numbers as representations of quantity and to investigate the relationship between the various components of number knowledge and counting.

This paper presents an overview of relevant literature on early number development and the relationships between various components of number understanding. The development of quantity understanding through counting will be discussed, followed by other components of number, including protoquantitive schema, subitizing and part whole understandings. A model is then proposed of how these aspects of number support each other in children's developing understanding of number as a representation of quantity.

Overview of Literature

Gelman and Gallistel (1978) suggest that children have an innate understanding of the principles needed to count; the one to one correspondence principle, the constant order principle, and the cardinal principle. While these three principles are all important, it is the cardinal principal that relates to the idea of quantity.

Counting and the Cardinal Principle

Gelman and Gallistel (1978) argue that children as young as two and a half years of age understand that the number said at the end of a counting sequence represents the number of items in the set. They claim, in effect, that if children are applying this principle, then they are showing they understand that numbers are representations of quantities. The question of whether children actually do understand the cardinal principle is therefore important.

On this matter, other researchers argue to the contrary that children as old as five years of age do not really understand the significance of the last word said in a counting sequence (Baroody & Wilkins, 1999; Bermejo, 1996; Bryant, 1997; Fuson & Hall, 1983; Nunes & Bryant, 1996). Such researchers are in general agreement that it is possible for children to repeat the last word of the counting sequence in response to a question asking "How many?" without really linking this to the idea that this number is telling them the quantity of the set. Baroody and Wilkins, for example, say, "initially, children may not realize that object counting serves the purpose of determining the number of items in a collection and may make no effort to remember their count" (p. 53). Bryant also reflects this view, when he says, it is "possible for a child to understand that the last number counted is the important one and still have no idea about its quantitative significance" (p. 59).

Ginsburg (1982) found that children often treated numbers as names or labels, rather than as signifying quantities. He cites an example of a child who counted a set of one blue and four red marbles. The blue marble was last in the line and so was the fifth in the sequence. From then on, the child called all blue marbles "five". Fuson and Hall (1983) found that some children respond to the "how many" question by recounting the items they just counted. They explained this by suggesting that children interpret the "how many" question "as a request for the counting act, rather than as a request for the information gained from the counting act" (p. 64). A study by Fuson and Mierkiewicz (in Fuson & Hall, 1983) found that some three and four year old children would end their counting with different number words on successive counts. When this was pointed out, the children consistently said the last number they had used was the correct one.

The above research suggests that children often do not apply the cardinality principle to a collection that they have just counted. Nunes and Bryant (1996) conclude, "… it is one thing to be able to count and answer the question 'how many' but quite another to understand the significance of the number uttered at the end of the counting as a measure of set size" (p. 41).

How then, can we know whether children understand the quantitative significance of the number said at the end of the counting sequence? Geary (1994), based on research by Wynn (1992), suggests that:

> One way … is to ask them to hand you two, three or seven items. Children who have a basic understanding of cardinality will count out the requested number of items and then hand them over. Children who do not understand cardinality usually grab a handful of items, without counting (p. 19).

Fuson (1988) (cited by Nunes & Bryant, 1996) reported that children in the five and six year age range were able to produce a set of a given number when asked. That is, they chose to count when asked to give the researcher a set amount. Geary says that some three year olds and many four and five year olds are able to use counting when they are asked to get an amount of objects in this way.

Nunes and Bryant (1996) argue that children only show full cardinality understanding when they know what counting is for and use it to solve problems, in particular, when they choose counting to match sets. They quote figures from Fuson (1988) indicating that many five and six year olds who are proficient at counting do not choose to use it when asked to produce equivalent sets. According to Nunes and Bryant, children initially understand number words and counting as a means of quantifying a single set, and later generalize this understanding to the point where they can use it to compare the size of two sets or to construct equivalent sets. As Nunes and Bryant suggest, this means children not only have to know how to count but also when it is appropriate to count. If children do not choose to use counting to solve problems which require them to determine and compare quantities, then they have not really understood the counting system.

Protoquantitive Schema

Resnick (1989) suggests that children develop a large store of quantity knowledge during their preschool years that forms the basis of their later mathematical development.

She terms this ‘protoquantitive’ as it consists of knowledge about quantities without the associated number words attached. Resnick links this knowledge with the children’s innate knowledge of number as described by researchers such as Huttenlocher, Jordan and Levine (1994) and Wynn (1995) and suggests that children begin to put words to the quantity knowledge that they had as infants.

According to Resnick, children develop three main protoquantitive schemas during their preschool years, comparison, increase/decrease and part whole schemas. While all these schemas are important, protoquantitive comparison is most directly linked to the number aspects included in this paper.

Resnick, Bill, Lesgold and Leer (1991) say that young children, before they are two years old, are able to compare sets and express quantity judgements and, soon after this, are able to use some associated quantity words. This means that they become able to respond to and use words such as “big”, “lots”, and “most” to make quantity comparisons. These comparisons are based on direct perceptual judgements rather than any form of measurement process. In a later publication, Resnick (1992) suggests that children’s protoquantitive comparison schema is their earliest form of mathematical reasoning and forms the basis for later numerical comparisons using numbers.

Geary (1994) does not use the term “protoquantitive schema” but discusses children’s understanding of order relationships. He refers to the research of Bullock and Gelman (1977) which found that more than 90% of the two years olds they studied were able to correctly use relational information to identify which of two sets contained either “more” or “less”. This adds weight to Resnick’s suggestion that children as young as two years of age have a protoquantitive comparison schema.

Resnick et al. (1991) use the research of others (for example, Sophian, 1987) to argue that children’s protoquantitive schemas exist as separate knowledge from their counting schema. Irwin (1996) seems to agree with Resnick et al., “My interviews with children confirm the presence of counting and protoquantitive concepts as separate areas of knowledge, understood separately before they are integrated into an ability to reason with numerical quantities” (p. 138).

Subitizing

According to Gallistel and Gelman, (1991) the term subitizing was first used by Kaufman et al. in 1949 to name the process used by adults to give rapid numerosity judgments for small arrays of simultaneously presented dots. Wynn (1995) defines it as “the ability to recognize small numbers of items automatically without having to engage in conscious counting” (p. 36).

Starkey and Cooper (1995) suggest that subitizing is fundamentally the same as the ability of infants to make judgments about numerosity. Other researchers, such as Geary (1994) and Wynn (1995), also conclude that there is a direct link between subitizing and infant abilities to judge numerosity. Wynn states that the upper limit of adult’s subitizing ability matches the upper limits of infant discrimination abilities and so concludes that the same quantification process underlies each.

Many researchers have found evidence of young children’s ability to subitize when they are not yet able to count. Starkey and Cooper (1995) found that two year old children subitized to correctly identify small amount of one, two, and three. They investigated the counting ability of this group of children and found that while some could use the one to one principle and the stable order principle, none were able to use the cardinal principle and thus could not be said to have “counted” to say how many.

Fuson and Hall (1983) found that children between the ages of two and five could readily distinguish one from two items and half of them could subitize three and four items although only one child they studied was able to count correctly. Wynn (1992, 1995) found similar results, for example, when children were asked to give the interviewer a number of items they tended to use subitizing and not counting for the smaller numbers. In this study, the children could count and yet preferred to subitize for the smaller numbers. Similarly, Sophian, Wood and Vong (1995), while investigating children's inferential understanding of number, found that three and four year olds could subitize to say how many in sets of two, three, five and six items (they did not use sets of one or four). Most of these children were also able to use counting and yet many chose to use subitizing for small sets. Finally, Starkey and Cooper (1995) claim that "Subitizing has developmental primacy in that it is present and producing representations of numerosity before the age at which verbal counting is used to produce representations of cardinal number" (p. 417).

It seems, therefore, that subitizing precedes the development of counting, but how are these seemingly different processes accommodated in the mind of a child? Starkey and Cooper suggest that subitizing and counting exist along side one another in a child's mind as distinctly different processes. They argue that subitizing is distinctly different from counting in that; (a) it is much faster; (b) children rarely make mistakes with small quantities; (c) when subitizing children may use the same number for different large quantities, for example, always use the number 10, whereas with counting they assign different numbers and think that both are right; (d) when subitizing sequential tagging and place keeping behaviours are not used, subitizing is a covert behaviour; and (e) the range of subitizing and counting expand at different rates. They also quote the work of Klahr and Wallace (1976) and Wynn (1992) to suggest that "subitizing is used to inject meaning into verbal counting" (p. 419). That is, it is only when children begin to link the final word in the counting sequence with the ability to subitize to say how many in the set, that they develop an understanding of the cardinality of the last word said in counting.

Labinowicz (1985) suggests that young children may initially be surprised when the final word in the count is the same as the number they have subitized and this provokes them to reflect on the use of the number words in the counting sequence. According to Starkey and Cooper (1995) it is this that leads children to understand the purpose of the counting process. Their research suggests that children typically begin to understand the cardinal word principle at about three and a half years and extend their subitizing range from three to four at about the same time. They suggest that this is when children begin to link subitizing and counting.

Clements (1999) quotes Baroody (1987) as suggesting that "subitizing is a fundamental skill in the development of students understanding of number" (p. 404). Certainly the research reported above indicates that subitizing is an important early mathematical process that develops informally and allows children to quantify sets well before they are able to use counting for this purpose. Children may develop the principles of counting along side their ability to subitize, initially not connecting the different uses of the number words. Subitizing may well be the foundation that children need to understand the quantitative significance of the sequence of words used in counting.

Part Whole Understanding

A number of researchers (Bobis, 1996; Fisher, 1990; Geary, 1994; Gray, 1998; Resnick, 1989; 1992; Ross, 1989) have suggested that understanding part whole relationships is an essential basis for understanding number generally. In her research, Fisher worked with kindergarten children and compared a curriculum that emphasized part whole relationships with a "normal" curriculum that emphasized counting based strategies. She found that the children in the part whole group were not only more successful on the part whole tasks, but also on those involving cardinality understanding. Bobis (1996), like Fisher, worked with kindergarten children and suggested that an emphasis on counting encouraged children to become over-reliant on counting strategies to solve even simple problems. She says that counting will not help children to see that, for example, the number five can be decomposed into three and two or four and one.

Resnick (1992) proposes that children's knowledge of numbers develops through a number of levels. Firstly, children learn to think protoquantitively, that is, they understand the relationships between quantities without the associated number words. Secondly, they learn to think about numbers as measures of quantities which describe a property of physical material. Thirdly, children become able to think of numbers as conceptual entities and can disassociate the numbers from the quantities that they represent. This suggests that children link the relational understanding of parts and wholes that they have developed, to their growing understanding of the number words. Later, they lift the numbers out of the quantitative contexts and are able to think about the additive composition of the numbers alone, without reference to the material. It is when children are able to think of numbers as conceptual entities that they understand the additive composition of numbers, without having to think about the size of a collection. In this way, children develop a deeper understanding of cardinality, they know that the number represents the total quantity of a set no matter which way it is displayed or partitioned. Without part whole understanding, cardinality resides in specific sets and counting whereas, with it, cardinality resides in the number no matter how it is displayed.

Synthesis of Literature

The research overviewed in this paper suggests that children will not develop an understanding of number as a representation of quantity through counting alone. This development is a complicated process that involves the interaction of a number of different quantitative aspects of a child's daily life. From the research discussed above, a model (see Figure 1) is proposed of how these different components interact to contribute to a child's developing understanding of numbers. The placement of the different components within the model indicates the interaction of the various components through time, not at particular times. Each is discussed briefly below.

Protoquantitive Comparison

Children, from about two years of age, become able to associate relational words with their innate ability to compare two amounts. They are able to say which is bigger or which amount has more or which has the most. Resnick calls this protoquantitive knowledge, though other researchers do not use this term, instead talking about relational knowledge.

Subitize

Children's ability to subitize small amounts seems to develop out of their early ability to compare quantities. Children learn to associate a particular number word with a particular quantity. Starkey and Cooper (1995) found that by age two most children in their study could subitize one, two and three items, at three and a half years of age children subitize up to four items and by five they subitize up to five items. Sophian, Wood and

Vong (1995), however, suggested that three and four year olds could subitize up to six items.

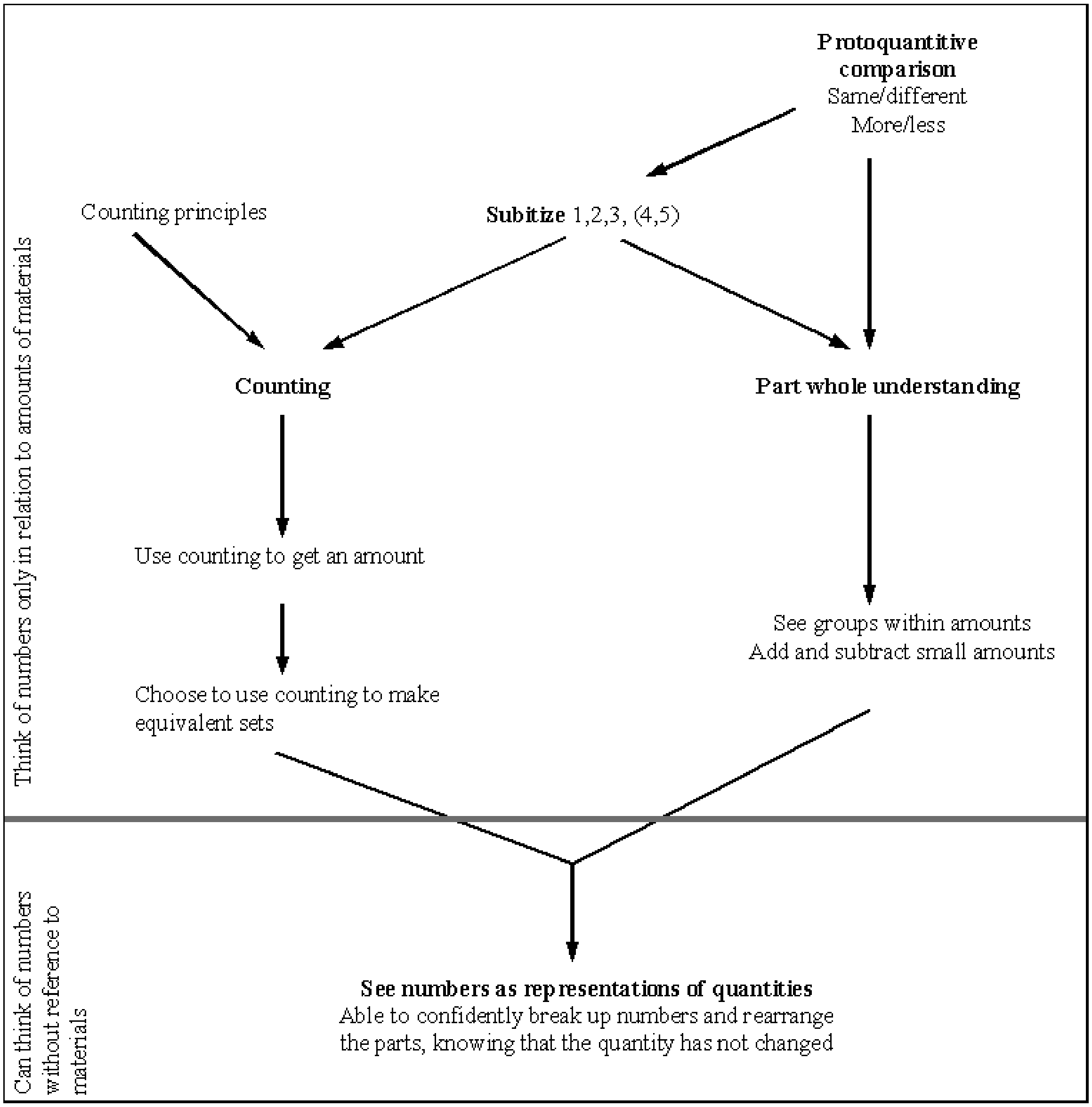

Figure 1. Children learning about number as a representation of quantity – A Model.

Counting Principles

At about two years of age, children begin to learn some of the principles of counting. They initially learn the first few words in the number sequence and to use one to one correspondence. Later they learn to give emphasis to the last word in the count. Children seem to initially learn these things as part of their socialization and may not link them with the idea of finding out how many. Even when children learn to repeat the last word of the number sequence in response to the how many question, they may not link this with the idea of quantity.

Use counting to get. Counting and subitizing initially exist along side one another in a child's mind as distinctly different processes. Children then begin to link the list of count words with the quantities that they know through subitizing and begin to understand that the last word said at the end of the count is telling them how many items in a collection. Children thus learn the quantitative significance of the number words in the counting process. According to Fuson (1988) (cited by Nunes & Bryant, 1996) children, at about five years of age are able to use counting to quantify single sets and to get an amount of items when asked.

Use counting to make equivalent sets. After children understand numbers words and counting as a means of quantifying a single set, they develop a trust in their counting processes and learn that no matter which way they count a collection they must always get the same result. As a consequence they "trust the count" and choose to use it to solve relational problems such as to make equivalent sets.

Part whole understanding. Children learn to connect the number words and quantity understanding they have from subitizing to part whole situations. This allows them to develop an understanding of the part whole relationships of numbers attached to particular quantities. They can see, for example, that five fish could be made up of a group of three fish and a group of two fish. This helps them to develop a more robust understanding of the numbers they use in counting. They come to trust that no matter which way a collection is arranged or partitioned, the quantity of the set will always remain the same.

See numbers as representations of quantities. Children's understanding of number from counting, subitizing, and part whole situations comes together so that they become able to think about numbers as representations of quantity. They are able to disembed the number from the situation and so become able to think of any five items as "five". The number becomes a conceptual entity in it's own right. They understand the additive composition of number and so can think of numbers as compositions of other numbers, the number five, for example, can be thought of as three and two. They can work with numbers alone without having to refer to a quantity of materials.

Conclusion

This model has been used as the basis for an investigation involving 25 children with learning disabilities in a Western Australian school (Treacy, 2001). Tasks were developed for each of the components listed above and these were used to individually interview the children. It was found that the children showed understandings similar to those suggested by the model above. For example, there were some children who showed no evidence of understanding the quantity aspect of counting and yet could subitize to three or more.

The teachers in this school found the model and the associated tasks particularly helpful in working out what their students knew and what they needed to know in order to develop a deeper understanding of number. Further research is needed however, to establish whether this model would be helpful for teachers working with children within the 'normal' range of intellectual ability.

References

Baroody, A. J., & Wilkins, J. L. M. (1999). The development of informal counting, number, and arithmetic skills and concepts. In J. Copeley (Ed.), *Mathematics in the early years, Birth to five* (pp. 48–65), Reston, VA: National Council of Teachers of Mathematics.

Bermejo, V. (1996). Cardinality development and counting, *Development Psychology,32*, 263–268.

Bobis, J. (1996). Visualisation and the development of number sense with kindergarten children. In Ed. J. M. Mulligan, & M. Michelmore (Eds.), *Children's number learning: A research monograph of MERGA/AAMT* (pp. 17–33). Adelaide: AAMT.

Bryant, P. (1997). Mathematical understanding in the nursery school years. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 53–67). East Sussex: Psychology Press.

Clements, D. H. (1999). Subitizing: What is it? Why Teach it?, *Teaching Children Mathematics, 5,* 400–405.

Fischer, F. E. (1990), A part-part whole curriculum for teaching number in the kindergarten, *Journal for Research in Mathematics Education, 21,* 207–215.

Fuson, K. C., & Hall, J. W. (1983). The acquisition of early number word meanings: A conceptual analysis and review. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 49–107). New York: Academic Press.

Gallistel, C. R., & Gelman, R. (1991), Subitizing: The preverbal counting process, In K. William, A. Ortony, & F. Craik (Eds.), *Subitizing: The preverbal counting process in memory, Thought and emotion* (pp. 65–81). Hillsdale, NJ: Hillsdale Press.

Geary, D. (1994). *Children's mathematical development: research and practical application,* Washington DC: American Psychological Association.

Gelman, R., & Gallistel, C. R. (1978), *The child's understanding of number,* Pennsylvania: Harvard University Press.

Ginsburg, H. (1982). *Children's arithmetic,* Austin, TX: Pro-Ed.

Gray, E. (1998), Compressing the counting process: Developing a flexible interpretation of symbols. In I. Thompson (Ed.), *Mathematics - Study & teaching (Primary)* (pp 63–72). Buckingham: Open University Press.

Huttenlocher, J. Jordan, N. C., & Levine, S. C. (1994). A mental model for early arithmetic. *Journal of Experimental Psychology: General, 123,* 284–296.

Irwin, K. C. (1996). Young children's formation of numerical concepts, or 8= 9 + 7. In H. P. Mansfield, N. A. Pateman & N. Bednarz (Eds.), *Mathematics for tomorrow's young children* (Vol 16, pp. 137–151).

Labinowicz, E. (1985), *Learning from children. New beginnings for teaching numerical thinking, A Piagetian Approach.* Sydney: Addison Wesley.

Nunes, T., & Bryant, P. (1996). *Children doing mathematics,* Oxford: Blackwell.

Resnick, L. (1989). Developing mathematical knowledge, *American Psychologist, 44,* 162–169.

Resnick, L. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday mathematical knowledge. In G. Leinhardt, P. Putnam, & R. Hattrup (Eds.), *Analysis of arithmetic for mathematics teachers* (pp. 373–429). Hillsdale, NJ: Erlbaum..

Resnick, L., Bill, V., Lesgold, S. & Leer, M. (1991). Thinking in arithmetic class. In B. Means, C. Chelemer, & M. Knapps, (Eds.), *Teaching advanced skills to at-risk students: Views from research and practice* (pp. 27 –67) San Francisco, CA: Jossey-Bass.

Ross, S. H. (1989). Parts, wholes, and place value: A developmental view. *Arithmetic Teacher* (February), 47–50.

Sophian, C., Wood, A. M., & Vong, K. I (1995). Making numbers count: The early development of numerical inferences. *Developmental Psychology, 31,* 263–273.

Starkey, P., & Cooper, R.G. Jr. (1995). The development of subitizing in young children. *British Journal of Developmental Psychology, 13,* 399–420.

Treacy, K. (2001). Children's developing understanding of quantity and number as a representation of quantity. Unpublished Masters Thesis, Murdoch University, Perth, Western Australia.

Wynn, K. (1992), Evidence against empiricist accounts of the origins of numerical knowledge. *Mind and Language, 7,* 316–332.

Wynn, K. (1995), Origins of numerical knowledge. *Mathematical Cognition, 7,* 35–60.