Year 8 Students' Reasoning in a Cabri Environment

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Year 8 students were introduced to the concept of mathematical proof before participating in conjecturing-proving tasks involving dynamic environments. This paper focuses on the argumentations of two pairs of students who completed a Cabri-based *Quadrilateral Midpoints* task. Supported by dynamic feedback and teacher intervention, even those students with limited understanding of geometric properties and relationships were able to engage in productive argumentation, conjecturing, and proving, with the dynamic geometric software serving as a cognitive bridge between empirical justification and deductive reasoning.

Following the demise of proof in school mathematics during the latter half of the 20th century, mathematics curricula in many countries now place increased importance on the need for students to justify and explain their reasoning. In the case of geometry, concern has been expressed, however, that dynamic geometry software, such as Cabri Geometry II™ and The Geometer's Sketchpad®, may be contributing to a data-gathering approach to school geometry, with empirical evidence becoming a substitute for mathematical justification. Noss and Hoyles (1996, p. 235), for example, assert that "school mathematics is poised to incorporate powerful dynamic geometry tools in order merely to spot patterns and generate cases." Used appropriately, though, dynamic geometry software can provide students with a rich visual environment for geometric reasoning.

Boero, Garuti, Lemut, and Mariotti (1996) suggest that students are more successful with proof construction if they are actively engaged in argumentation during which they formulate their own conjectures. Boero et al. use the expression *cognitive unity* to describe the continuity which they claim must exist between conjecturing and successful proving. Boero (1999) asserts that the nature of arguments used by students depends on the establishment of a culture of theorems in the classroom, on the nature of the task, and the specific kinds of reasoning emphasised by the teacher.

The Study

The research reported here is part of a broader study of conjecturing and proving involving student argumentations in dynamic environments (see Vincent & McCrae, 2001; Vincent, Chick, & McCrae, 2002). This section of the research focuses on the influence of students' understanding of geometric properties and relationships on their ability to engage in deductive reasoning, and the role of feedback in a dynamic geometry environment in supporting students' reasoning.

The participants in the research were 29 Year 8 students at a private girls' school in Melbourne in 2001 who had been selected for an extension mathematics class on the basis of their performance in Year 7 mathematics and their non-verbal reasoning ability.

Research Design

Pre-testing. A van Hiele test (C. Lawrie, personal communication, 1/5/1997) was used to assess the students' initial understanding of geometric properties and relationships.

Pre-requisite geometry. Prior to commencement of the lessons associated with the research, geometry that would be required in the conjecturing and proving tasks was taught or revised—properties of triangles and quadrilaterals; angles associated with parallel lines cut by a transversal; Pythagoras' theorem; and similar and congruent triangles. During these lessons, the students used Cabri to investigate angle properties of triangles, including exterior angles; properties of quadrilaterals, including diagonal properties; and angles associated with parallel lines cut by a transversal. The emphasis in each of these exploratory activities, however, was on empirical data and identifying properties, with no reference to why these properties were true. This was important so that conjecturing and argumentation in geometry, and the concept of proof, would be new experiences for the students in the lessons associated with the research.

Conjecturing-proving lessons. The students were introduced to the need for geometric proof, then simple proofs—for example, the proof that the sum of the angles of a triangle is 180^{o}—were discussed and modelled by the teacher-researcher. Following this introduction, the students worked in pairs on a number of conjecturing-proving tasks, most of which involved dynamic environments. These lessons, in which the students engaged in argumentations, were video-recorded and transcribed. Interventions by the teacher-researcher (indicated by TR in the transcribed argumentations) were a key feature of the argumentations, and served several functions, including clarification of the content of the students' statements, probing the meaning of statements, answering queries, correcting false statements, and re-directing the students' thinking if they had reached an impasse. The most important purpose of the teacher-researcher mediation, however, was to ensure that the students' arguments were based on sound mathematical logic.

This paper will focus on the argumentations of two pairs of students—Anna and Kate, and Jane and Sara—in an exploratory task using Cabri Geometry. The students had completed a paper-and-pencil *Joining midpoints* task, in which they were asked to prove that the segment joining the midpoints of two sides of a triangle is parallel to the third side. *Quadrilateral Midpoints* (see Figure 1) was a Cabri conjecturing-proving task designed to follow the *Joining Midpoints* task.

Quadrilateral Midpoints
Construct a quadrilateral in Cabri and label it *ABCD*.
Use the midpoint tool to construct the midpoint of each side of the quadrilateral and label the midpoints *P*, *Q*, *R*, and *S*.
Join the four midpoints to make another quadrilateral *PQRS*.
Make a careful ruler and pencil diagram of your screen construction.
Drag the quadrilateral *ABCD* and make a conjecture based on your observations.
Prove your conjecture.

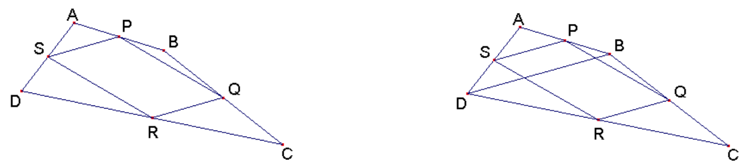
Figure 1. Cabri *Quadrilateral Midpoints* task.

Results and Discussion

Although the 29 Year 8 students represented the upper 25 percent of their year level, the van Hiele test responses indicated that they were not exceptional in their geometric understanding. Anna and Kate were generally confident with properties and relationships of simple geometric figures: squares, right-angled triangles, isosceles triangles, parallel lines, similar triangles, and congruent triangles. By contrast, Jane and Sara were uncertain of properties and lacked understanding of relationships between properties.

When given the *Quadrilateral Midpoints* task, Anna and Kate constructed and labelled the Cabri figure (see Figure 2a), and immediately conjectured that *PQRS* was a parallelogram, without dragging the construction to check.

01 Kate: It's a parallelogram [*PQRS*].
02 Anna: Yeah … we think it's a parallelogram.
07 Kate: Um … if there's a way we could prove that angle [*PQR*] is equal to that angle [*RSP*] …
08 Anna: Yeah.
09 Kate: And … then we can prove it's a parallelogram.



(a) (b)

Figure 2. Anna and Kate's Cabri quadrilateral construction.

At first Anna and Kate focused on angles:

10 Anna: Oh, maybe … [Anna drags point *B*] … no matter where the points go … wherever you move it that's [*PQRS*] always the same.
11 Kate: All we've got to prove is that that angle [$\angle SPQ$] and that angle [$\angle QRS$] are equal and that [$\angle PSR$] and that [$\angle RQP$].
12 TR: What was that?
13 Anna: Well, we have to prove that $\angle SPQ$ is equal to $\angle QRS$ and that $\angle PSR$ is equal to $\angle RQP$.
14 TR: And what other things do you know about parallelograms?
15 Kate: The sides.
16 Anna: The lengths of these [opposite sides of *PQRS*] should be the same … It all makes sense but …

Kate's rapid progress to a solution, shown below, suggests that she related the construction to the triangle midpoints proof that she had just completed, although she did not immediately refer to this. Even before constructing the diagonal *BD* (see Figure 2b), Kate seemed to have a mental image of the proof and omitted all the intermediate steps, although she later included these in her written proof. Implicit in Kate's verbal reasoning is the fact that if two segments are each parallel to a third segment, then they must be parallel to each other. Anna had not made the link with the triangle midpoints proof, and at first displayed doubt about Kate's suggestion to draw the diagonal *AC*:

17 Kate: If we draw a line from *B* to *D* and make a triangle … then they're [*S* and *P*] the midpoints of the two sides.
18 Anna: Mmm.
19 Kate: Then those two [*SP* and *DB*] will be parallel.
20 Anna: But if you put an extra line in here [from *D* to *B*] you're just making another parallelogram.
21 Kate: No, look! Look! [drawing the segment *DB*]

When Kate drew Anna's attention to the *Joining Midpoints* worksheet, Anna recognised the relevance of this proof, and that construction of the diagonal *AC* would enable them to prove that the other two sides (*PQ* and *SR*) of *PQRS* are also parallel:

22 Anna: But …
23 Kate: If you didn't have those two lines [*SR* and *PQ*] it would be the same shape as this [pointing to the *Joining Midpoints* worksheet] so *SP* is parallel to *DB*.
24 Anna: Mmm.
25 Kate: And then … it's just the same for that [pointing to *QR*]. See you've got the two triangles, one there, one there [*ADB* and *BCD*], so *PS* is parallel to *BD* which means it's parallel to *QR* …
26 Anna: And then if you do another one that way [indicating *AC*] …
27 Kate: Yeah.

In their written proof (Figure 3), Anna and Kate justified each statement and ordered their statements logically. They demonstrated a clear understanding of deductive reasoning and of the necessary and sufficient conditions for a quadrilateral to be a parallelogram.

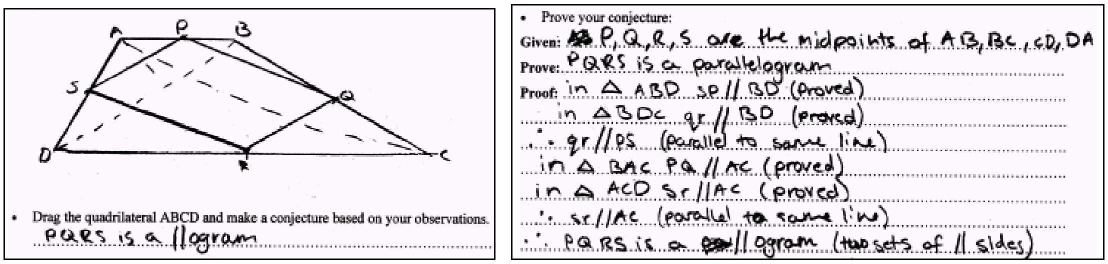


Figure 3. Anna and Kate's written proof for the *Quadrilateral Midpoints* task.

Unlike Anna and Kate, who noticed the parallelogram immediately, Jane and Sara focused initially on the triangles surrounding the parallelogram (see Figure 4), conjecturing that the triangles were similar or congruent. Their reasoning tended to be visual (for example, turn 13: "it's just flipped over"), or based on their recently acquired knowledge of congruent and similar triangles, so that they failed to notice other features of the figure.

01 Sara: Are those triangles there [ΔPBQ and ΔQCR] similar to those [ΔPAS and ΔRDS] or something like that?
02 TR: Drag it around and see what you think.
03 Jane: Which ones?
04 Sara: Is that [ΔPBQ] similar to that [ΔQCR] and that [ΔRDS] similar to that ΔPAS]?
05 Jane: But you can change the shapes.
06 Sara: I think they are.
07 Jane: Can this [ΔQCR] be similar to that [ΔRDS]?
08 Sara: Congruent … I reckon they're congruent.
09 Jane: They're not. [They continue dragging the quadrilateral.]
10 Jane: Mmm …
11 Sara: Those two are the same [ΔQCR and ΔRDS]. They both [*RS* and *RQ*] go from the same midpoint, they both share that line [DC] …
12 Jane: That [*DR*] and that [*RC*] are equal … yes, they are …
13 Sara: It's just flipped over … just as if they've gone on top of each other …
14 Jane: They are … just because they are … that's [*SD*] the same as that [*QC*] and that's [*DR*] the same as that [*RC*] and the angle in between … no it won't …
15 Sara: It is, it is … if you flipped that over it would be just the same.
16 Jane: Measure angle.
17 Sara: No, it's not the same now … if you flipped that over … do you want to do that? Can it be like that? … oh, no they're not …
18 Jane: Look, that one's [$\angle QCR$] an acute angle and that one's [$\angle SDR$] obtuse.
19 Sara: When they're added together they equal … something. Are we getting close? Is it anything to do with triangles?

Jane and Sara not only failed to notice that *PQRS* was a parallelogram, but even when their attention was drawn to it, they were unable to identify it correctly:

20 Jane: I was thinking, could the inside quadrilateral be similar to the outside quadrilateral?
21 Sara: No … that's got nothing to do with it. Look at that … you can go like that [Sara drags point A] and it's nothing like it.
22 Jane: That's true. [They drag *ABDC* again] […]

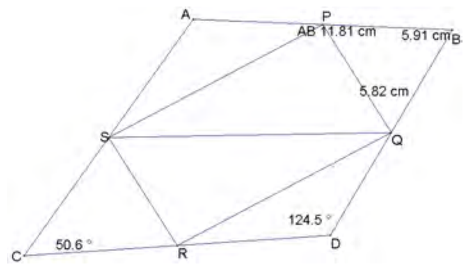


Figure 4. Jane and Sara's Cabri construction for the *Quadrilateral Midpoints* task.

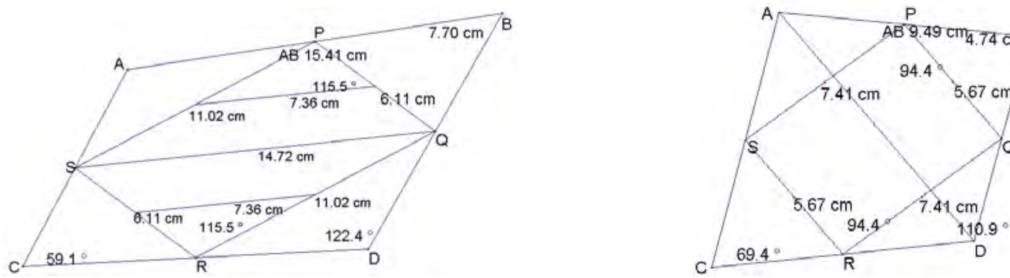
27 Sara: Can I draw a line somewhere? [Sara draws the segment *SQ*] Are those two congruent? … those two trapezia? [*ABQS* and *CDSQ*].
28 Jane: They're similar.
29 Sara: Similar [they continue dragging the *ABDC*].
30 Sara: They're the same.
31 Jane: No, they're not.
32 TR: What about the quadrilateral *PQRS*?
33 Jane: It's a rectangle [Jane drags *ABDC* again].
34 Sara: It's a square!
35 Jane: It's regular.
36 TR: What's regular about it?
37 Jane: The sides.
38 TR: One of you said it's a rectangle and one of you said it's a square. Keep dragging and watch it.
39 Sara: It's a rectangle! [*PQRS* does not appear to be a rectangle] It's a rhombus!
40 Jane: You can't prove that it's a rhombus.
41 TR: What do you know about the sides of a rhombus?
42 Sara: They're all the same.
43 TR: Do they look the same?
44 Tog: No.
45 TR: What do a rectangle, a square and a rhombus have in common?
46 Sara: Parallel sides. It's a parallelogram!
47 Jane: Parallelogram!
48 TR: Well, we could check it in Cabri and then we could try to find a proof. What would you expect to find if it was a parallelogram?
49 Sara: That the sides were the same. [Sara measures the other three sides of *PQRS*]
50 Sara: [Sara measures *SR*] 5.01
51 Jane: [pointing to *PQ*] 5.01!
52 Jane: [Sara measures *QR* and *PS*] Yes! It's a parallelogram!
53 TR: Is it always a parallelogram?
54 Sara: [Sara drags point *S*] Yeah, it's always a parallelogram.

Sara then attempted a visual explanation:

55 Sara: I think it's … um … because the midpoint always stays the same and if the angles of the triangle are always joined to the shape …
56 TR: Which triangle?
57 Sara: I mean of the square … sorry … of this … the parallelogram … this parallelogram is always … it's centred … it's in the very centre of the whole shape because of the lines … therefore it stays there. It always stays in the middle …

Following my suggestion that they could add construction lines, Jane then seemed to recognise that the previous triangle midpoints proof might be useful, but instead of constructing a diagonal of the quadrilateral *ABDC*, she was misled by the diagonal, *SQ*, that they had drawn previously. She joined the midpoints of the sides of *PQRS* (see Figure 5a), measured the segments, and found them to be equal:

58 Jane: Can you do the midpoint of that triangle. [Sara constructs midpoints of *PS* and *PQ*]. And then join them.
59 Sara: And shall we measure them?
60 Jane: Yep.
61 Jane: Yes! [Excitedly as she notices that the segments joining the midpoints are equal]
62 Sara: We already knew they were congruent triangles though.
63 TR: How do you know that?
64 Sara: Because it's side angle side.
65 Jane: But we don't know …
66 Sara: But that line's [pointing to the segment joining the midpoints of *PS* and *PQ*, and *RS* and *RQ*] the same on both … so wouldn't that mean? If those two lines were the same distance apart on both that would mean the angles were the same … is that right?
67 Jane: [Hesitatingly] Yeah …
68 Sara: Mmm … I'll measure those angles then.
69 Sara: Same.
70 Jane: Same, yeah … so … ooh … that's the same as that [pointing to the small triangles formed by joining the midpoints]



(a) (b)

Figure 5. Jane and Sara: Adding construction lines

It seemed that Jane and Sara may be thinking of the *Joining Midpoints* task completed two days ago, but they were unable at this stage to relate the diagram from that task to their *Quadrilateral Midpoints* diagram. Sara was uncertain of the properties of parallelograms, and had also become confused about what they were trying to prove:

71 Sara: Yeah, and they're the same [pointing to ΔPSQ and ΔSRQ]. So would that mean those two were the same two? [pointing to ΔAPS and ΔCSR] Oh, no … so that … oh … so would these two be the same? … oh, no … […]
73 Jane: Yeah. That's [*SQ*] parallel to them … [pointing to the segments joining the midpoints] […]
77 Jane: Don't we want to do it that way? [moves her finger between *P* and *R*]
78 Sara: Diagonals intersect at right angles … or whatever … What are we trying to prove?
79 Jane: We're trying to prove that *PQRS* is a parallelogram … and is always a parallelogram.
80 Sara: What are the … um … the …
81 TR: Properties?
82 Sara: Yeah, properties … what are the properties of a parallelogram? I've forgotten.
83 Jane: The lengths [of the opposite sides] are always the same.
84 Sara: Exactly the same length.
85 TR: Another property would be?
86 Sara: Opposite angles … are equal … and we've got both …
87 TR: Only by measurement though. We haven't actually proved it.

As Jane and Sara seemed unable to continue, I suggested that they should delete *SQ* and the two segments joining the midpoints of the sides of *PQRS*, and look back at the *Joining Midpoints* proof. Jane now recognised that if she drew the diagonal, *AD*, of quadrilateral

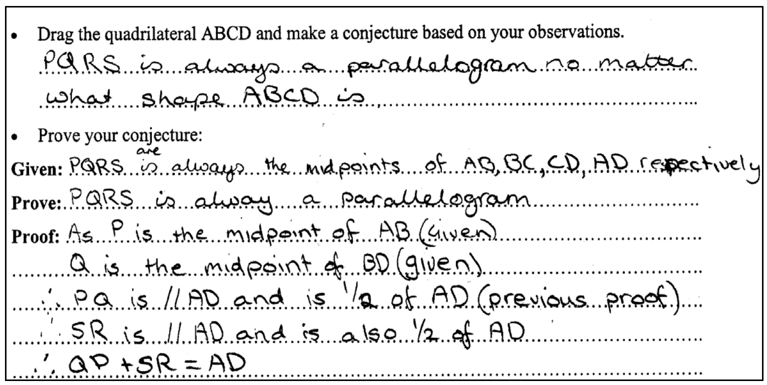
ABDC (see Figure 5b), she could prove that *PQ* was parallel to *SR*, her satisfaction evident in her exclamation: “Clever me!”.

88 Jane: Oh! … I’ve got it! [draws segment *AD*] ‘cause this [*PQ*] and this [*SR*] is the same, this [*PQ*] is the midpoint line so this [*PQ*] is parallel to that [*AD*] and that’s [*PQ*] parallel to that [*SR*].

89 TR: Well done!

90 Jane: Clever me!

Although Jane and Sara commenced their written proofs appropriately, they seemed to lose sight of what they were actually trying to prove, with Jane concluding that *QP+ SR =AD* (Figure 6). The proofs were incomplete, and showed a limited of awareness of the necessary and sufficient conditions to prove that *PQRS* was a parallelogram.



- Drag the quadrilateral ABCD and make a conjecture based on your observations.

PQRS is always a parallelogram no matter what shape ABCD is.

- Prove your conjecture:

Given: PQRS is (are) always the midpoints of AB, BC, CD, AD respectively

Prove: PQRS is alway a parallelogram

Proof: As P is the midpoint of AB (given)

Q is the midpoint of BD (given)

∴ PQ is // AD and is ½ of AD (previous proof)

∴ SR is // AD and is also ½ of AD

∴ QP + SR = AD

Figure 6. Jane’s written proof for the *Quadrilateral Midpoints* task.

Discussion and Conclusions

Jane and Sara’s argumentation contrasts sharply with that of Anna and Kate, who recognised the parallelogram immediately, with Kate quickly making the connection with the previous triangle midpoints proof. During the course of the research, it was apparent that several of the Year 8 students were developing this ability to recognise particular configurations which they had encountered previously. Koedinger and Anderson (1990) assert that this ability to skip steps in reasoning by parsing a geometric diagram into subfigures, or ‘perceptual chunks’, is characteristic of high school geometry proof experts.

The ability of Jane and Sara to conjecture and argue was handicapped by their lack of confidence with quadrilateral properties and relationships, and with appropriate geometric language. Despite the dynamic nature of the imagery provided by Cabri, it was some time before Jane and Sara recognised the invariant parallelogram, despite their dragging of the quadrilateral. Duval (1998), reporting on a study with 13–14-year-olds in a pencil-and-paper environment, notes that some students failed to recognise relevant subfigures within a figure, with the result that they did not make a connection with the appropriate theorem. Laborde (1998) suggests that dynamic geometry computer environments focus students’ attention on invariant properties which they may not notice in a static drawing, and it is the recognition of these invariant properties that provides access to the underlying geometry. As seen with Jane and Sara, however, even in a dynamic geometry environment it cannot be assumed that students will notice invariant properties. Having convinced themselves that the task was about similar or congruent triangles, Jane and Sara had focused on the triangles surrounding the parallelogram, and failed to notice the parallelogram. Their poor understanding of properties of special quadrilaterals no doubt contributed to this lack of observation, but there was also a tendency for Jane and Sara to draw hasty conclusions based on visual evidence and to focus on their most recently acquired knowledge—similar and congruent triangles.

By contrast, Anna and Kate saw the parallelogram immediately, even before they dragged the figure, and Kate was able to relate the figure to the previously completed triangle midpoints task. Although eventually Jane was also able to relate the figure correctly to the *Joining Midpoints* proof (turn 88), unlike Kate, at first she could not visualise which construction lines were required, reflecting her poorer understanding of relationships within the figure. Anna and Kate were better able to articulate their justifications so that when they began to construct their written proof, they had already acquired a strong sense of logical order for the steps of deductive reasoning. Statements which had been implied in the verbal reasoning were now made explicit.

Anna and Kate were cognitively ready for engaging in deductive reasoning, and in the relatively simple context of the *Quadrilateral Midpoints* task, were not dependent on dynamic feedback. Jane and Sara, by contrast, were more dependent on teacher intervention and on the dynamic feedback from the software. Even before Jane and Sara noticed the invariant parallelogram, dragging and Cabri measurements provided the reinforcement and contradiction to allow them to progress in their argumentation, and in their geometric understanding. Despite their limited geometric knowledge, engagement in the argumentation enabled Jane and Sara to formulate a valid conjecture, and their confidence in the empirical feedback provided the motivation to construct a proof. Although their argumentation was substantially longer than that of Anna and Kate, and their proof was incomplete and imperfect, Jane and Sara were acquiring the beginnings of an understanding of deductive reasoning, as well as a strong sense of satisfaction. Rather than eliminating the students' need for proof, the dynamic geometric software had served as a cognitive bridge between empirical justification and deductive reasoning.

References

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof*. July/August. http://www-didactique.imag.fr/Preuve/Newsletter/990708Theme/990708ThemeUK.html (30/8/2001).

Boero, P., Garuti, R., Lemut, E., & Mariotti, A. M. (1996). Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 113–120). Valencia, Spain: PME.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century* (pp. 37–52). Dordrecht, The Netherlands: Kluwer.

Koedinger, K., & Anderson, J. R. (1990). Abstract planning and perceptual chunks: Elements of expertise in geometry. *Cognitive Science*, *14*, 511–550.

Laborde, C. (1998). Visual phenomena in the teaching/learning of geometry in a computer-based environment. In C. Mammana & V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st century* (pp. 113–121). Dordrecht, The Netherlands: Kluwer.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Mathematics Education Library, Vol. 17. Dordrecht, The Netherlands: Kluwer Academic Press.

Vincent, J., & McCrae, B. (2001). Mechanical linkages and the need for proof in secondary school geometry. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4 (pp. 367–374). Utrecht, The Netherlands: PME.

Vincent, J., Chick, H., & McCrae, B. (2002). Mechanical linkages as bridges to deductive reasoning: A comparison of two environments. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4 (pp. 313–320). Norwich, UK: PME.