What Do They See When They Look? Student Perspectives on Equations

David Godfrey
The University of Auckland
<godfrey@math.auckland.ac.nz >

Michael O. J. Thomas
The University of Auckland
<m.thomas@math.auckland.ac.nz>

For some time it has been known that student use and knowledge of the constituent parts of equations, have not mirrored those of mathematicians. This paper describes some understandings of particular parts of the object "equation" displayed by lower secondary school students and seeks to analyse them in terms of properties of those parts of equations, including the equals sign. We find that students display a number of different conceptions of what an equation is, and this appears to be connected with their perspective on the role of operators and transitivity. This data has assisted with construction of a framework for understanding the mathematical equation object.

Equations are ubiquitous in mathematics and understanding them forms a crucial part of the early school mathematical experience. While to the experienced mathematical eye they appear as a single object they are composed of a number of separate entities. Each of these parts, and, indeed, the gestalt they comprise, may be viewed from several perspectives according to Laborde (2002). The views include a surface or perceptual one and a mathematical one, whereby the mathematical properties of the entity or object are understood. In this paper we address the hypothesis that for a mathematical equation, it is the arithmetic numbers, the symbolic literals, the operators, and the '=' symbol that each hold mathematical properties that subsequently contribute to the whole equation. Hence understanding the mathematical equation object would require the formation and integration of these individual properties. The hypothesis that the binding agent for understanding these constituent parts is language is not addressed here but is deemed crucial by the authors.

While many students develop a reasonable working knowledge of arithmetic numbers and their operators, the same cannot be said of symbolic literals (Küchemann, 1980). Some learning environments encourage a process-oriented view of mathematics (Thomas, 1994) where the object of study is not cognitively engaged, and hence pseudo-conceptions (Vinner, 1997) are more likely to occur. Once these pseudo-conceptions are in place (such as letter as object) they can be very resistant to change and may act as cognitive obstacles when a student is encouraged to perceive a mathematical object, such as an equation, via its properties.

There is clear evidence that students exhibit problems going beyond the separator operator interpretation of the equals symbol (Baroody & Ginsburg, 1983). The study by Denmark et al. (1976) reports that first grade students were able to develop some flexibility in accepting the use of the equals symbol in a variety of arithmetic sentence structures (e.g., $3 = 3$, $3 + 2 = 4 + 1$, $5 = 4 + 1$), achieving this by means of balancing activities and corresponding written identities. However the students still viewed the equals symbol primarily as an operator rather than a relational symbol. Herscovics and Kieran (1980) investigated student acceptance of an equivalent amount written in a variety of ways (e.g., $4 + 7 = 12 – 1$) and found that they were able to accept and work quite comfortably with arithmetic identities containing multiple operations on both sides. It appears that the meaning of the equals symbol needs to evolve from the intuitive ideas of sameness or counting the total found in arithmetic, and the idea of the result of a procedure (Kieran, 1981), to a notion of the equivalence of algebraic statements with reflexive, symmetric and transitive properties. While this process of change does not appear to come easily or quickly to many students, Whitman and Okazaki's (2003) results show that student understanding of "=" as equivalence could be improved from the first to second grade.

When we ask the question 'What is an equation?' we may still get a number of differing responses. In their research with preservice teachers, Hansson and Grevholm (2003) found that very few preservice teachers wrote that $y = x + 5$ was an equation, tending to a numerical interpretation of $y = x + 5$. One view of an equation is as a structural statement or representation of a mathematical relationship between entities that are the objects of an algebraic and/or arithmetic system. One of the requirements for generating and adequately interpreting an equation structurally is a conception of the reflexive, symmetric, and transitive character of the equals sign as an equivalence relation. It is through these properties that the equals symbol conveys the concept of equivalence. Collins dictionary (Borowski & Borwein, 1989, p. 194) deals with the definition in this way:

> Equation, *n*. a formula that asserts that two expressions have the same value; it is either an *identical equation* (usually called an IDENTITY), which is true for any values of the variables, or a *conditional equation*, which is only true for certain values of the variables (the ROOTS of the equation).

Thus they specify two possible types of equation: a *conditional equation;* and an *identical equation* and, for example, $2x + 1 = 6$ would be a *conditional equation,* but $2(2x + 1) = 4x + 2$ would be an *identical equation*.

5. Pick out those statements that are equations from the following list and write down why you think the statement is an equation.
   a) $k = 5$
   b) $7w - w$
   c) $5t - t = 4t$
   d) $5r - 1 = -11$
   e) $3w = 7w - 4w$

8. For what values of x does: (show any working)
   a) $x + 2 = 2 + x$
   b) $x + 1 = 2 + x$
   c) $x + 3 = x + a$

10. If $p = q + 3$ and $q + 3 = 2 - r$, write p in terms of r.

11. If $a = 2 - b$ and $c = 2 - b$, write a in terms of c.

Figure 1. Some of the questions from the questionnaire.

We may ask, what are the distinctive properties of each of these types of equation that construct the mathematical object? The purpose of this present research study was to investigate younger students' perceptions of equation and to relate them to the constituent parts of the mathematical object.

Method

The research reported on here forms part of a larger cross-sectional study considering students' understanding and use of the equals sign. In this study 29 Form 4 students (age 14-15 years) from a large, coeducational, high socio-economic school in Auckland were given a questionnaire with 12 questions aimed at different aspects of equation we had previously identified (Godfrey & Thomas, 2003). They were given 55 minutes to complete the questions, doing so in a normal mathematics class. There were 8 female and 21 male students. Figure 1 contains a summary of the questions considered in the analysis below.

Results

Data from a previous part of this research with 81 Year 12 students (15-18 years old) on how students understand the components of an equation, particularly the variables and the equals sign, enabled us to begin the process of constructing a framework attempting to map out perspectives on equations. This framework (Godfrey & Thomas, 2003, see Figure 3) suggested that properties of the = sign, symbolic literals, numbers, and operations all contribute to an overall perspective on equation. In this present study, question five of the questionnaire (see Figure 1) asked students to identify from a list of five statements those that they thought were equations, giving reasons for their choices. Most of the student responses fell into one of the three distinct categories, as described in Table 1.

Table 1
Categories of Responses for Question 5

Statements which are an equation	Stated reason for the choice	Number of students in category
a, c, d, e	Needs an = sign	8
b, c, d, e	Needs an operation to carry out	3
c, d, e	Needs and = sign and an operation to carry out	9

These categories may be exemplified by the responses of students 7, 1, and 8 respectively, as shown in Figure 2. It seems that the category 1 students are basing their decision solely on the surface structure of the equation; if it contains an equals sign then it is an equation. They responded:

7. An equation has an = sign in it

12. Equation, = is present

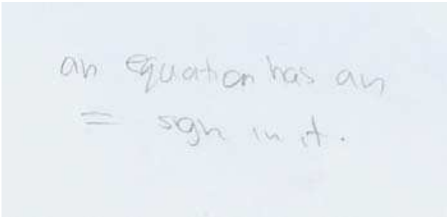
13. because they have equals in them

In contrast, the category 2 students have the perspective that as long as there is an operator present and they are able to carry out an operation to produce a result then it is an equation, although there may be no = sign present. They no doubt think that it is not a problem to supply the sign themselves prior to writing the answer (Kieran, 1981). Here are the ways in which this was explained by these students:

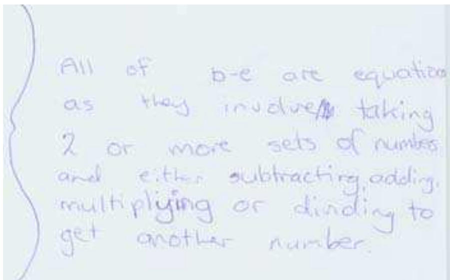
1. …they involve taking 2 or more sets of numbers and either subtracting, adding or multiplying or dividing to get another number.

16. because you have to subtract, add, multiply and/or divide.

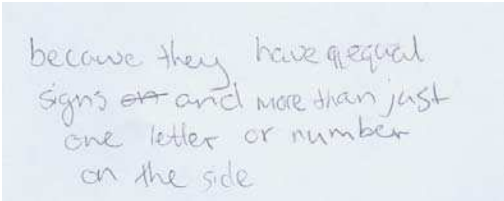
26. because it envolves [sic] –, +, ×, ÷ sign in it.



Student 7



Student 1



Student 8

Figure 2. Examples of each category of equation response in question 5.

The category 3 students have a subtly different perspective from the second. Since for them the = sign has to be present initially, but there must still be an operator, or as student 8 puts it “more than just one letter or number on the side”, implying an operator between them. They comment that:

8. because they have equals sign and more than just one letter or number on the side.
11. because they all involve =s & –s.
14. because it is something you must work out to find a value.
16. because it has an answer and a means of getting the answer.
24. because it has an answer and uses subtraction.
29. They all have an = sign and it’s not just a statement like a).

Among those not fitting neatly into this classification were 6 students (there were also 3 no-response students) who gave a mixture of answers with little discernible pattern. It seems that they may be in transition between the identifiable groupings we have identified, or they may have developed pseudo-conceptions. What some have in common though is an emphasis on the need for operators, and on wanting to ‘solve’ and equation. Their reasons were:

Student 3 – gave c and e, “Because it has a correct answer”; “the answer is right.”

Student 4 – gave b, d and e, “yes, because w stands for a number and you are minusing it from $7w$”; For b he said “no, the t’s do not represent a number…”, so he has a specific unknown view but only of particular letters.

Student 6 – gave c since “question in correct order and is correct. He displays a process-oriented perspective, and a need for a correct answer in the equation.

Student 20 – gave b, c and e “because there is still stuff to figure out. She is close to a group 2 member but she only applies the operator to letters and not to part d which has $5r – 1$.

Student 22 – gave a, c, and e as equations because “everything is balanced and works out [correctly].” He seems to exclude d because it is a conditional equation, not an identical equation like c and e, but accepts part a since k=5 can be seen as completed. This is an unusual stance since the 'standard' type of equation students of this age are used to solving would be of the conditional type d.

Student 27 – gave c and d, saying "has equal sign and only one unknown number" stating for e that "it is already true so it is not an equation." He seems to be moving toward category one, but also wants to have some work to do. His failure to group e with c may be a misread or an inconsistency.

Our previous model of equation understanding based on work with older students had identified the importance of the properties of the = sign, and the student perspective on the use of letters as symbolic literals. We were interested to know how the categories above would mesh with these understandings, and so began an analysis of responses to questions 8 and 10 (see Figure 1). Question 8 required an understanding of the use of letter as variable, since one needed to be able to say that the equation was true for all (real) values of x, or in part c) that it was only true when a=3. Question 10 requires the ability to see and use the transitivity property of the = sign in an equation. Table 2 shows the results of the 20 students in the 3 categories above on questions 8, 10 and 11 (see Figure 1).

Table 2

Results on Questions 8, 10 and 11 for the 3 Categories of Students

Category of Equation	No. with Q8 correct	No. with Q10 correct	No. with Q11 correct
1	a 8 b 5 c 5	6	8
2	a 1 b 1 c 1	1	1
3	a 5 b 3 c 2	3	5

Category 2 was a small sample and student 14 got all three questions correct while students 1 and 26 got nothing correct, and made only 2 responses between them, making them the least successful group. Overall, it appeared that the category 1 students were generally more successful than the category 3 group of students, with 5 of the 8 getting all three questions completely correct. The summary of the individual responses below (Table 3) shows that students 11 and 14 from category 3 were able to answer all the questions, while students 8 and 9 got both transitivity questions correct but were unable to use a knowledge of variable to answer question 8. In contrast, student 29 in this group had a knowledge of variable but not of transitivity. Only student 20 of the 6 non-categorised students correctly answered question 10, and only student 27 from this group got any part of question 8 correct (he was totally correct).

What do these results tell us? It may be deduced that the students who still had some view of equation as requiring operators and solutions did not perform quite as well as those who used the surface structure of the presence of an = sign. This latter group may well have subsumed much other knowledge of equations under this umbrella catch-all, since they could mostly cope with letter as variable and the transitive property, and an interview would reveal useful information here.

Table 3
Summary of the Individual Responses of Some Students

Category 1	Q8c)
	7. NR; 12. if a equals 3 then all values, if not no values; 13. x = anything if $a = 3$; 15. x = r (any real number) but only if $a = 3$; 17. all values; 18. anything as long as a is three; 21. x can equal any real number; 30. depends on what a is.
	Q10
	7. $p = 2–r$; 12. $p = 2–r$; 13. $p = 2–r$; 15. $p = –r+2$; 17. $p = 2r$; 18. $p = 2–r$; 21. $p = 2–r$; 30. $q = 2–r–3$, $q = –1–r$, $p = –r+2$.
Category 3	Q8c)
	2. Any –3; 8. NR; 9. Any; 11. Any; 14. all when $a = 3$, none when a = greater than or less than 3; 23. $a = 3$; 24. NR; 25. NR; 29. Any when a equals 3, and none when a is anything else.
	Q10
	2. $p – 2 = r$; 8. $p = 2–r$; 9. $p = 2–r$; 11. $p = q–2+r$; 14. $p = 2–r$; 23. NR; 24. NR; 25. NR; 29. $r = –q–1$, $q+1 = –r$.

Discussion

Some objects in mathematics are purely theoretical, with no physical counterpart, while others do have such a counterpart. For example, Laborde (1995, 2002) has discussed with reference to geometry the nature of the difference between a drawing and a figure. She explains that the former is physical and perceptual, while the latter is theoretical and mathematical. However, most, if not all mathematical objects have symbolisms, which may be viewed in different ways. A procept is one such variation on perspective, the process versus object view of a symbol. But this is not the only way we can see symbols. When we perceive the symbolisation of an object we may simply have a surface or observational view (Thomas, 2001), but in order to get a mathematical perspective of what it represents we have to interpret what we see (c.f., Booth & Thomas, 2000). This interpretation of the symbolisation or representation often requires us to interact with the object (Thomas & Hong, 2001), giving rise to identification of the object's properties, often underlying invariants. One way that properties, and hence mathematical objects, arise is by reflective abstraction, and we then synthesise these abstracted properties into a new object, a mathematical one. For example, when first learning geometry, we may be told that a figure is a rectangle and our conception will be based on properties obtained by perception, or by observation (Thomas & Hong, 2001). Even when we have seen many rectangles, we will not have constructed the mathematical object of rectangle. It is only when we have the properties that constitute the object that we have constructed the mathematical entity.

Consider the corresponding situation with regard to equation. We can form a surface recognition of an equation based on the surface observation that it contains an '=' sign, as in this study. This may be based on a understanding of properties of equations, such as the transitive nature of the = sign, and the use of letter as variable that we found. However, it may also be a pseudo-conception lacking depth of understanding, revealed by questions about whether $x = x$, $4 = 5$, $y = f(x)$, etc. are equations. The equation conception can be elusive, not easy to tie down in terms of the properties that define the mathematical object. What our research so far suggests is that the gestalt mathematical equation object comprises arithmetic numbers, variables, operators, the equals sign and the structure combining them. This structural view is no doubt both fed by the developing understanding of properties of the others parts, and in turn feeds back to further understanding of the object. One's mathematical understanding of these constituent parts becomes welded into a coherent whole, the mathematical equation that is much greater than the sum of the parts. Of course, our mathematical understanding is also mediated by language and so any framework should include its vital role, and we are investigating the binding influence of this, and the structural perspective. The evidence here is that students may have quite well developed understanding on one or more of the constituent parts, with little in other areas, although, of course, there will be crossover effects between them.

Conclusion

We have concentrated in this paper on how students decide on what is, or is not an equation, and their ability to use transitivity of the equals sign, and to cope with letter as variable. Previously (Godfrey & Thomas, 2003) we have provided more evidence for the role of student perspectives on variable and the equals sign in this process, and have distinguished a variety of perspectives on equation. While there is certainly much more to be said about equation conceptions, these are summarised in our provisional outline framework for equation in Figure 3. We note that one reason students may lack certain perspectives is that teaching may not highlight these properties explicitly. Thus, students will not be able to interact fully with the mathematical equation object. Certainly, there is effort required to assist students to enrich their perspective on equation.

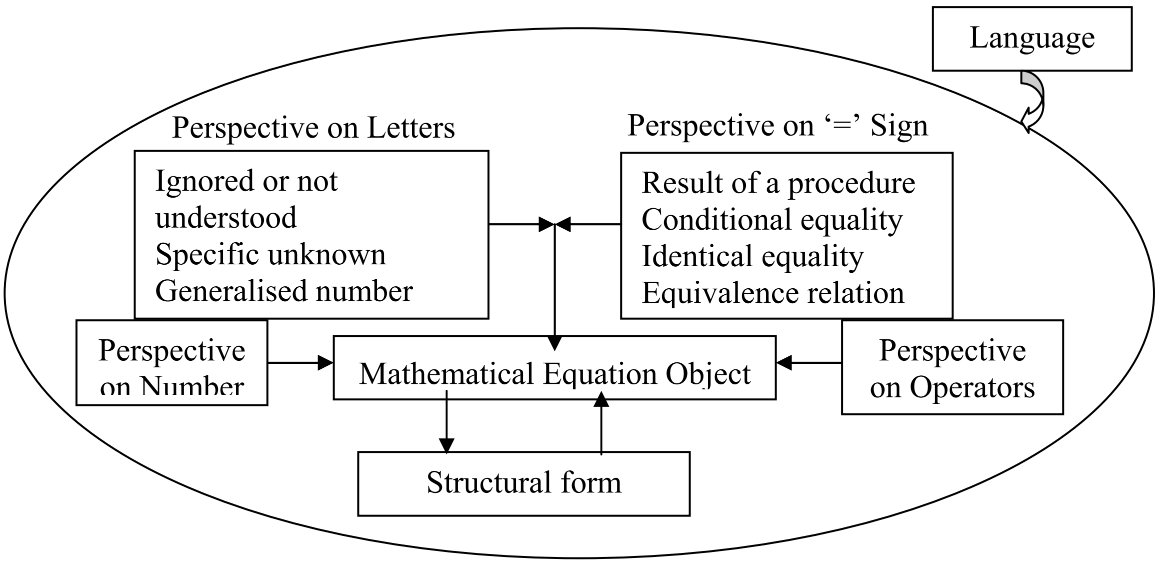


Figure 3. An outline framework of the mathematical equation object.

References

Arcavi, A. (1994). Symbol sense: Informal sense-making in formal mathematics, *For the Learning of Mathematics*, *14*(3), 24–35.

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the equals sign. *Elementary School Journal, 84*(2), 199–212.

Booth, R. D. L., & Thomas, M. O. J. (2000). Visualisation in mathematics learning: Arithmetic problem-solving and student difficulties, *Journal of Mathematical Behavior, 18*(2), 169–190.

Borowski, E. J., & Borwein J. M. (1989). *Dictionary of mathematics*. London: Collins.

Denmark, T., Barco, E., & Voran, J. (1976). *Final report: A teaching experiment on equality,* PMDC Technical Report No. 6, Florida State University. (ERIC Document Reproduction Service No. ED144805).

Fey, J. (1990) Quantity. In Steen, L.A. (Ed.) *On the shoulders of giants: New approaches to numeracy*. (pp. 61–94). National Academy Press, Washington, D.C.

Gattegno, C. (1974). *The common sense of teaching mathematics*, Educational Solutions, New York.

Godfrey, D., & Thomas, M. O. J. (2003). Student perspectives on equation: Constructing the mathematical object, In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.) *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th Mathematics Education Research Group of Australasia Conference, Geelong, pp. 396-403). Sydney: MERGA.

Hansson, O., & Grevholm, B. (2003). Preservice teachers' conceptions about $y = x + 5$: Do they see a function? *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education*, Honolulu, Hawaii, *3,* 25–32.

Herscovics, N., & Kieran, C. (1980). Constructing meaning for the concept of equation, *The Mathematics Teacher, 73*, 572–580.

Kieran, C. (1981). Concepts associated with the equality symbol, *Educational Studies in Mathematics*, *12*(3), 317–326.

Küchemann, D. E. (1981). Algebra, in K.M. Hart (Ed).*Children's understanding of mathematics: 11-16* (pp. 102–119). John Murray.

Laborde, C. (1995). Designing tasks for learning geometry in a computer-based environment: The case of Cabri-géomètre, In L. Burton & B. Jaworski (Eds.) *Technology: A bridge between teaching and learning mathematics* (pp. 40–68), London: Chartwell Bratt.

Laborde, C. (2002). The process of introducing new tasks using dynamic geometry into the teaching of mathematics B. Barton, K. C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.) *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, Auckland, pp. 15–33). Sydney: MERGA.

Mason, J, (1987). What do symbols represent? In Janvier, C. (Ed.) *Problems of representation in the teaching and learning of mathematics*. Hillsdale, NJ: LEA.

Tall, D. O., Thomas, M. O. J., Davis, G., Gray, E., & Simpson, A. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior, 18*(2), 223–241.

Thomas, M. O. J. (1994). A process-oriented preference in the writing of algebraic equations. In G. Bell, B. Wright, N. Leeson, & J. Geake (Eds.), *Challenges in mathematics education: Constraints on construction* (Proceedings of the 17th Mathematics Education Research Group of Australasia Conference, Lismore, Australia, pp. 599-606). Lismore: MERGA.

Thomas, M. O. J. (2001). Building a conceptual algebra curriculum: The role of technological tools, In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.) *International Congress of Mathematical Instruction (ICMI) The Future of the Teaching and Learning of Algebra Proceedings* (pp. 582–589). Melbourne: The University of Melbourne.

Thomas, M. O. J. & Hong, Y. Y. (2001). Representations as conceptual tools: Process and structural perspectives, *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. *4*, pp. 257–264). Utrecht, The Netherlands: PME.

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics*, *34*, 97–129.

Whitman, N. C., & Okazaki, C. H. (2003). What "=" means, *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 262). Honolulu, Hawaii: PME.