Operating with Decimal Fractions as a Part-Whole Concept

Kathryn C Irwin
University of Auckland
<k.irwin@auckland.ac.nz>

Murray S. Britt
Auckland College of Education
<m.britt@ace.ac.nz>

Most research on both the assessment and teaching of decimal fractions has dealt with the decimal fractions as static units, not ones involved in operations that involve compensation. This study examined 1356 students' ability to add, subtract, multiply and divide numbers that included a decimal portion using the appropriate compensation for each operation. A fifteen-item test was given to 12-year-old students that assessed both their ability to use compensation and decomposition with whole numbers and with numbers that included a decimal fraction. Results showed that if students were able to use the appropriate compensation with whole numbers, between 47% and 64% were also able to use compensation to solve problems dealing with numbers that included a decimal fraction. This indicated that these students understood the part-whole nature of decimal fractions. It is suggested that while compensation has not been used to teach decimal fractions in our schools, it provides an additional way of enabling students to operate with decimals in ways similar to that used by people outside of school.

Decimal fractions continue to perplex many students and adults, as evidenced by studies such as that of Stacey, Helme, Steine, Baturo, Irwin & Bana. (2002). The many misconceptions that students and adults hold are the result of inappropriate generalisation from whole numbers, from concepts that are appropriate for fraction, or from confusion of the number line and place value columns so that decimal values are seen as negative numbers and the decimal point as the same as 0. Yet despite the fact that adults may err in tests of understanding of decimals as static units, most people do operate with them appropriately in common aspects of their lives. They operate by rounding or truncating them to the nearest number that they can easily understand. They accept that the price of petrol at $\$1.10^{9}$ per litre is really $1.11 and a currency exchange of $0.5423 to €1 can be truncated to about 54 or even 50 cents to the Euro. In places where a 15% tip is expected, people learn that instead of multiplying the cost by 0.15 you take a tenth of an amount of the bill and then add half of that tenth to give 15%. In countries where the currency unit is of small value (e.g., the Philippines), people tend to ignore the smaller units and pay only in rounded larger unit. In these situations people deal with decimal fractions flexibly and rarely think of them as detached from whole numbers. They are more likely to think of them as small parts of a whole unit, which is exactly what we would like students to appreciate.

Often our teaching and assessment emphasise the decimal units as a static unit in isolation, largely because they are the new units being taught. They are taught and tested as static units that need to be named or ordered. Assessments of students' understanding of decimal fractions in this country may require students to order decimal by size from smallest to largest (e.g., 0.45, 0.5, 0.465). Another common assessment, as used in Martine and Bay-Williams (2003) and Hart, Brown, Kerslake, Kuchemann, & Ruddock (1985) requires students to place a decimal such as 0.7 on a number line, or to give a decimal that comes between 0.45 and 0.46. Baturo & Cooper (1997) reported a study in which students were asked to re-unitise tenths as hundredths. The assessment that accompanies the New Zealand Numeracy Project (http://www.nzmaths.co.nz/Numeracy/Index.htm) asks students to indicate what 37.5% is as a decimal. The research of Resnick, et al. (1989) and of Stacey et al. (2002) deals only with the relative size of decimals, not their meaning when combined with whole numbers or when operated upon. All of these assessment tasks treat decimals as static units, not as numbers that are operated with, usually not in relation to whole numbers.

Much of the research literature on teaching students about decimals has separated decimals from meaningful contexts, often in order to teach students about place value, an abstract, albeit important, aspect of decimals. Programmes such as that of Hiebert and Wearne (1988) and Swan (1983) have attempted to make the size of the relative units apparent to students. Relatively little has been written about students' ability to operate with decimals dynamically in a manner that demonstrates that they understand the meaning of decimals. One exception to this is the work of Irwin (2001).

This research looked at students' ability to operate with numbers that include a decimal fraction using the same principles of compensation or distribution that would enable them to work out whole number problems mentally. This would be similar to the way in which adults might mentally calculate that 15% of a price as in the example of the tip, above, or know that if one needed three pieces of cloth that were 1.9 metres long that would be the same as 3 times 2 metres minus 3 times 0.1 metres, or 5.7 metres.

The data presented here were collected as part of a study that explored students' ability to recognise appropriate compensation and decompensation manipulations and use them in working with numbers. This was dependent on their number sense of the parts that numbers can be split into (eg, Beishuizen, 1993; Reys et al., 1999).

Method

Participants and Procedure

All 1356 year 8 students (aged 12 years) in six intermediate schools were assessed. These students took the test described below in one period of their normal mathematics class time. Three of the schools were participating in the New Zealand Numeracy Project and three schools were not participating in this project. Each of these two subgroups comprised three schools identified as Decile 1, 3, and 5. Deciles are rankings that roughly equate to socioeconomic status of the parents. They are based on the statistical likelihood of students passing the now-outdated school examination taken by students at age 15. Decile 1 is the lowest, and includes students from backgrounds similar to those who were statistically least likely to pass the examination, (Dialogue Consultants, 1990). The New Zealand Numeracy Project encourages the use of mental manipulation of numbers in a way that requires understanding that numbers are made up of parts. However it does not teach any of the modifications in this test with decimal fractions.

Materials

A test of students' ability to generalise the strategies of compensation and distribution in computation was especially written for this evaluation. This test required students to appreciate the validity of breaking numbers into parts for accurate operation with them. It also required them to appreciate which manipulations of numbers were appropriate for different operations.

The test had five pages or sections, each of which required a different manipulation. Each page started with an example of a child using the appropriate manipulation and was followed by three problems for students to work out. Section A covered compensation in addition and is shown in a condensed format in Figure 1. Section B covered compensation in subtraction, with the exemplar child doing 87 - 48 by changing it into 89 - 50 and 183 - 97 changed into 186 - 100. Section C involved distribution in multiplication, with the exemplar child working out 3 × 58 by doing (3 × 60) - 6 and 4 × 96 as (4 × 100) - 16. Section D involved compensation in multiplication with the examples of 48 × 5 being equivalent to 24 × 10, and 36 × 25 being equivalent to 9 × 100. Section E covered compensation in division with the exemplars being 160 ÷ 5 being equivalent to 320 ÷ 10 and 300 ÷ 25 being equivalent to 1200 ÷ 100.

On each page, the first two problems involved whole numbers with the second problem including larger numbers than the first, and the final problem involving numbers with a decimal fraction to one place (tenths). The intention was to provide for increasing levels of complexity that would lead to outcomes that reflected the multi-structural, relational and extended abstract levels of SOLO taxonomy (see Biggs & Collis 1982; Biggs, 1995). The same presentation format was used for each section. Figure 1 shows a condensed version of the first page of this assessment. Adequate space was provided for students to show their working.

Jason uses a simple method to work out problems like 47 + 25 and 67 + 19 in his head

Problem	Jason's calculation
47 + 25	50 + 22
67 + 19	66 + 20

Show how Jason's method works for

36 + 48

Show how Jason's method works for

268 + 96

Show how Jason's method works for

35.8 + 4.6

Figure 1. Example of demonstration item and problems for students in this assessment.

This report refers largely to the last question in each section: that which required students to carry out the demonstrated manipulation with numbers that included tenths. These problems are given in Figure 2.

Section A	35.8 + 4.6
Section B	47.2 – 6.7
Section C	4 × 7.8
Section D	48 × 0.5
Section E	31 ÷ 0.5

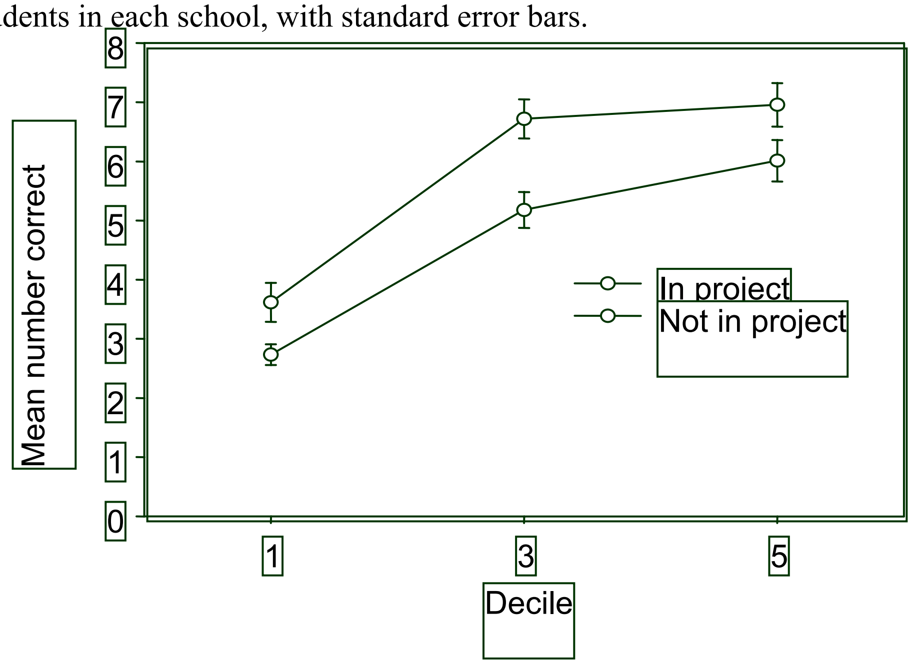
Figure 2. Example of demonstration item and problems for students in this assessment.

Analysis

Each answer was marked as correct or incorrect depending upon the method used to solve it. For example, in the first problem, 36 + 48 in Figure 1, students would be marked as correct if they turned the problem into 34 + 50 or 40 + 44. They would not be credited if they wrote a vertical algorithm and proceeded in the traditional manner, as this did not provide evidence that they understood the critical aspects of the exemplar calculation. Minor calculation errors were ignored. However, a misplaced decimal point or answer that was wrong by a factor of ten was not considered a minor problem. Such errors were seen as evidence of not understanding the principle as applied to numbers with a decimal fraction.

Overall Results

While there were some students in each school who showed that they understood the examples and could follow them, the overall analysis showed that students who were in schools within the New Zealand Numeracy Project were significantly more successful on this test than were similar schools that were not in the project ($F(1,976) = 19.00$, $p < 0.001$). There was also a significant effect of decile ($F(1,975) = 63.11$, $p < 0.001$), but there was not a significant interaction between these two factors ($F(1,976) = 0.65$, $p = 0.52$). That can be interpreted as indicating that each school in each decile group benefited from involvement in the project, but schools from different decile groups did not differ significantly in the amount that they benefited. (The above results are based on the Brown-Forsythe analysis, rather than the standard analysis of variance, because the variances of the groups were significantly different). Figure 3 shows the mean score of students in each school, with standard error bars.



. *Figure 3.* Mean scores and standard error bars of schools whose students took the test of generalisation of strategies.

When mean scores for each item were compared, students in the project were more successful than were students from similar deciles but not in the project on every item. The project students would have had opportunities to devise and experiment with several flexible mental operational strategies but would not have had experience with all of the manipulations in this test. To our knowledge, their experiences would have been limited to working with whole numbers as decimals are treated separately from whole numbers in this project. It was therefore experimental on our part to see if students were able to extend their flexibility in the use of these strategies from whole numbers to decimal fractions.

As with the overall results, some students both receiving and not receiving the Numeracy Project were successful with the decimal items. However, a higher percentage of students receiving the Numeracy Project were successful on decimal items, as is shown in Table 1. It is our belief that the students receiving the Numeracy Project were able to transfer their understanding of manipulation of the parts of whole numbers to manipulation of decimal fractions.

The range of common errors that were shown indicated that some students who understood the whole number problems failed to extend that understanding to similar operations with decimals. Some students added, subtracted, multiplied or divided only the whole number portion of the numbers given and ignored, or did not change the decimal fraction portion (tenths). Others treated the decimal portion as a whole number. For example, some students turned 48×0.5 into 24×10. Other students turned this same problem into 24×0.10.

We were interested in the proportion of students who used the appropriate strategy on at least one of the whole number problems and who also used it on the task involving decimals. We felt that this value would provide a reliable measure of students' facility to extend understanding of operational strategies with whole numbers to those involving decimal fractions. This information is given in Table 1.

Table 1.
Facility of students in and not in the Numeracy Project who were successful in applying operational strategies to the whole number and decimal items.

Sections	Percentage of students successful on at least one whole number item		Percentage of students successful on the decimal item		Percentage of students successful on at least one whole number item also successful on the decimal item	
	Project	Non-project	Project	Non-project	Project	Non-project
A	78	64	44	33	57	51
B	37	28	24	15	64	54
C	49	40	18	13	36	32
D	55	39	30	18	54	47
E	43	32	24	18	57	57

The proportions of students both in the project and not in the project who were able to extend their facility with at least one whole number item to the decimal item are shown as a graph in Figure 4.

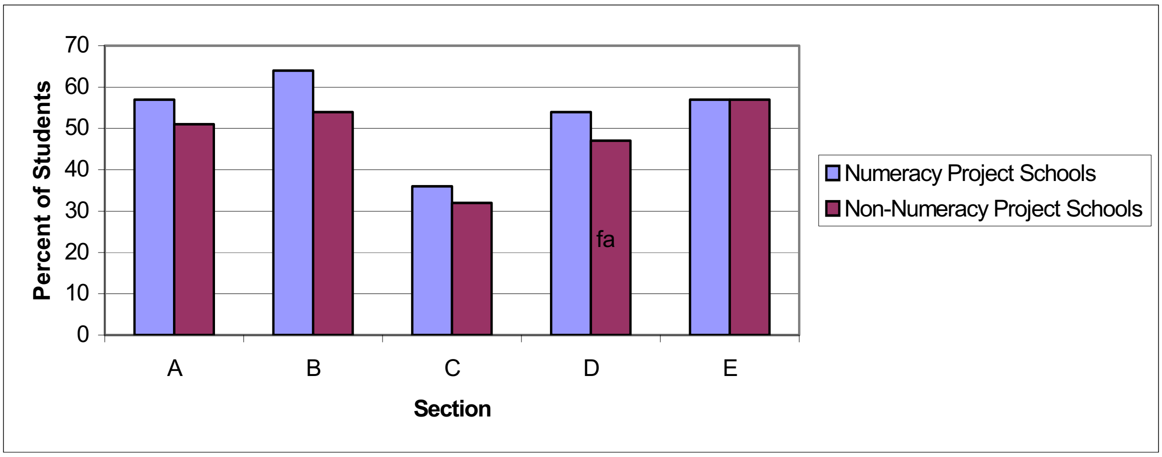


Figure 4. Percentage of those students who were successful on one or more whole number items who were able to extend that strategy for use with decimals.

Figure 4 shows that while the facility for extending whole number understanding to decimal understanding with these operations consistently favours those students who participated in the Numeracy Project, a relatively large percentage of both groups that could understand compensation and decomposition with whole numbers also could demonstrate it with numbers including decimal fractions. With the exception of items in Section C, between 47% and 64% of the students who were successful on at least one of the whole number problems was also successful with the decimal items.

Discussion

This study showed that some students were able to use part-whole manipulations in operating with numbers including tenths, while not having been taught to do this, as far as we know. Students in schools that were involved in the Numeracy Project and had been encouraged to think of the component parts of whole numbers for operating efficiently did better than students who were not involved in this project on both whole number manipulations and decimal manipulations. However, all students who understood breaking whole numbers into parts were more likely to be able to use this skill with decimal fractions. Another way of saying this is that being in a school using the Numeracy Project is a help in understanding part-whole relationships that can be used to manipulate decimal fractions, but it is not essential.

Students from all schools did best on problems that included decimal fractions on the four subtests involving different types of compensation: Sections A, B, D, and E. This required them to understand that, for example, 0.8 and 0.2 made up 1.0. It also required them to appreciate that appropriate compensation was different in addition and subtraction, and different in multiplication and division. They had more difficulty with the multiplicative distribution in Section C. This distribution requires more steps. For example, in many of the errors on this section students chose an appropriate larger or smaller number to multiply the larger number by but then failed to also multiply the number to be subtracted. For 8×79 they would write $(8 \times 80) - 1$, or for (3×298) they would write $(3 \times 300) - 3$. When a decimal fraction was added to these necessary manipulations, the students also had to note that 7.8 was close to 8, multiply 4 times 8, and then figure out that 0.2 had to be multiplied by 4 to subtract 0.8 to get the answer. Only 18 percent of Numeracy Project students and 13 percent of students not on the Project succeeded with these decimal operations (see Table 1). The greater cognitive load required of this task may account for the limited success on Section C items.

The data on compensation with numbers that included tenths from these four sections involving compensation can be viewed pessimistically or optimistically. From the pessimistic, or jar-half-empty, point of view, it is disturbing that, for any of the sections, less than 50 percent of these 12-year old students understood the part-whole nature of decimals well enough to successfully carry out the compensation tasks (see the middle columns of Table 1). Moreover, on Sections B, D, and E, the facility on any decimal task fell to 30 percent or lower. From the optimistic, or jar-half-full, point of view, the results show that slightly more than 50 percent of the students who understood manipulations involving the part-whole nature of whole numbers could generalise appropriate part-whole compensation strategies involving decimal fractions. To our knowledge, the students had not received any tuition with the part-whole compensation of decimal fractions.

Decimals, like common fractions, are the result of division, and are necessarily although not obviously a concept that requires an understanding of the relation of parts to a whole. The step to part-whole thinking is a major step forward from strategies that depend on some version of counting. This part-whole thinking with whole numbers can be seen in students aged 7 or 8 years. In New Zealand, we have tended to defer the introduction of decimals until students are about 10 years old. Many of these students are at a stage where their numerical thinking allows them to use part-whole concepts when they are able to apply compensation adjustments to the calculation of 2- and 3-digit whole-number addition and subtraction problems.

Conclusions and Implications

In this study we found that Numeracy Project students were better equipped to operate with decimals using a variety of part-whole strategies extended to decimals even though they had only been exposed to these strategies as part of their work with whole numbers. The question arises therefore as to what would be the effect of introducing manipulation with decimals soon after students showed confident understanding of the part-whole nature of whole numbers?

This is not the current approach to teaching decimals either in the Numeracy Project or in other teaching. As indicated above, there is currently no provision within the Numeracy Project materials for the use of part-whole strategies for operating with decimals (http://www.nzmaths.co.nz/Numeracy/2004numPDFs/Book%207%20Fract-Dec.pdf). The results from this study have led us to postulate that once tenths are understood as a tenth part of one whole, students' knowledge and understanding of decimals could be considerably strengthened by their experimenting with mental operations using the compensation ideas that were used to assess understanding in this study.

Understanding of decimal fractions will probably continue to be difficult no matter what methods are used to teach and assess them because of the inappropriate generalisations that students can make on the basis of prior knowledge. Several methods have been proposed and trialled to improve understanding. Moss and Case (1999) have suggested starting with percentages and moving from there to decimals. In the Numeracy Project, some lessons focus on the place value of decimals, for example using Unifix cubes in groups of 10 or pipes of different length which is an approach that originated at the University of Melbourne (Archer & Condon, 1999). Other suggested lessons base learning of decimal fractions on a number line that has previously been used for common fractions that are not base 10.

In our view no one approach to understanding decimal fractions is going to be student-proof. The approach taken in this assessment could form the basis of a teaching approach for students who understand the part-whole nature of whole numbers which quickly follows a sensible introduction and consolidation of the concept of decimal tenths.

References

Archer, S., & Condon, C. (1999). *Linear arithmetic blocks: A concrete model for teaching decimals*. Melbourne: Department of Science and Mathematics Education, University of Melbourne.

Baturo, A. R., & Cooper, T. J. (1997). Reunitising hundredths: Prototypic and nonprototypic representations. In E. Pehkonen (Ed.) *Proceedings of the 21st annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 57-64). Lahti, Finland: PME.

Beishuizen, M. (1993), Mental strategies or models for addition and subtraction up to 100 in Dutch second grades. *Journal for Research in Mathematics Education, 24*(4)*,* 294-323.

Biggs, J. (1995). Assessing for learning: Some dimensions underlying new approaches to educational assessment. *The Alberta Journal of Educational Research*, *41*(1), 1-17.

Biggs, J. B., & Collis, K. F. (1982). *Evaluating the quality of learning the SOLO Taxonomy*. New York: Academic Press.

Dialogue Consultants. (1991). *Development of socio-economic indicators of educational disadvantage in schools: Prepared for the Ministry of Education.* Auckland NZ: author.

Hiebert, J., & Wearne, D., (1989). Cognitive changes during conceptually based instruction on decimal fractions. *Journal of Educational Psychology, 81,* 507-513.

Hart, K., Brown, M., Kerslake, D., Kuchemann, D., & Ruddock, G. (1985). *Chelsea diagnostic tests.* Windsor: NFER-Nelson.

Irwin, K. C. (2001). Difficulties with decimals and using everyday contexts to overcome them. *Journal for Research in Mathematics Education, 32,* 399-421.

Martine, S. L., Bay-Williams, J. M. (2003). Investigating students' conceptual understanding of decimal fractions using multiple representations. *Mathematics Teaching in the Middle School, 8*(5)*,* 244-247.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122-47.

New Zealand Ministry of Education (2004). *New Zealand Numeracy Project*. Retrieved 24 March 2004 http://www.nzmaths.co.nz/Numeracy/Index/htm.

New Zealand Ministry of Education (2004) *Book 7: Teaching fractions, decimals and percen*ts. Retrieved 24 March 2004 http://www.nzmaths.co.nz/Numeracy/2004numPDFs/Book%207%20Fract-Dec.pdf

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education, 20,* 8-27.

Reys, R., Reys, B., McIntosh, A., Emanuelsson, G., Johansson, B., &Yang, D. C. (1999). Assessing number sense of students in Australia, Sweden, Taiwan and the United States, *School Science and Mathematics, 99*(2), 61-66.

Stacey, K., Helme, S., Steine, V., Baturo, A., Irwin, K., & Bana, J. (2001). Pre-service teachers' knowledge of difficulties in decimal numeration. *Journal of Mathematics Teacher Education, 4,* 205-255.

Swan, M. B. (1983). *Teaching decimal place value.* Nottingham: Shell Centre for Mathematics Education.

Acknowledgements

This research was funded by the New Zealand Ministry of Education. The views expressed are those of the authors.