Students' Views on Using CAS in Senior Mathematics

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Interviews were held with five volunteer students in the first cohort to use CAS in end-of-school mathematics examinations in Victoria, to review their experiences and the use they made of CAS. Four students were positive, and one negative. Students were highly reflective about their learning, discussing the need for mathematical understanding to use CAS. Students generally chose to use CAS to do questions quickly, or when by-hand skills were not trusted, or to find equivalent forms matching multiple choice answers in exams.

This paper reports on interviews held with volunteer students in the first cohort of year 12 Mathematical Methods (CAS) in Victoria, Australia. The subject MM(CAS) was established by the Victorian Curriculum and Assessment Authority (VCAA, n.d.) in conjunction with the CAS-CAT research project (CAS-CAT, n.d). It was the first mathematics subject where students were allowed to use CAS in all externally set examinations (VCAA, 2002). The content was based on that of the VCAA subject Mathematical Methods, a functions and calculus course where graphics calculators are permitted in the examinations. This is the main subject used for entrance to scientifically-based tertiary courses in Victoria.

The purpose of the interviews was to record students' experiences and to find out how they had used CAS in the examinations, looking especially at their appreciation of the need for algebraic insight (Pierce and Stacey, 2004a) and to what extent they exhibited effective use of CAS (Pierce and Stacey, 2004b). Teachers from the three pilot schools invited all MM(CAS) students to return to school after the final examinations in November 2002 for interviews with the researchers. Five students agreed: Ann and Dana from one school, Cate and Eve from a second school, and Beth from the third school. These are pseudonyms and gender may have been changed. The students were asked about their general impressions of studying MM (CAS) and about their use of CAS in examinations; they demonstrated how they had done some questions and identified questions that illustrated particular points. Each interview was conducted by two researchers, audiotaped and transcribed. Interviews had no bearing on students' grades.

This paper will discuss decisions students made about using CAS or not in the examinations and the way in which they went about learning mathematics and solving problems with CAS available. This report focuses on the symbolic aspects of CAS (i.e. the features beyond those of graphics calculators). During interviews students commented on a range of issues including the use of by-hand and CAS techniques for solving problems, how they learned to use CAS during the year and how unexpected CAS outputs were dealt with. The interviews revealed that the five students had a diversity of mathematical abilities and beliefs about use of CAS for mathematics, and so made an interesting sample. Ann was a very good mathematics student, with good by-hand skills. She enjoyed using CAS, and made both functional and pedagogical use of it. Dana had strong by-hand skills and preferred to solve problems by hand than with CAS. Beth had low confidence in her by-hand skills and in CAS and did not feel it helped her solve problems more quickly. Both Cate and Eve experienced some difficulties with mathematics, but confidently used CAS to

compensate for inadequate by-hand skills. They both stated that they would have enrolled in an easier mathematics stream if CAS had not been available as support.

> "I wouldn't have done [Mathematical] Methods if we didn't have these [i.e. CAS calculators]. I would have done General or Further [i.e. an easier Year 11 and 12 mathematics stream]". (Eve)

The fact that two of the five students said they would have enrolled in an easier mathematics stream without CAS can not be taken as indicative of the general situation: all the teachers reported a few students like this in their classes, but they are over-represented here due to interviewing volunteers.

By-Hand or By-CAS

All students had views about solving problems using by-hand or CAS methods in exams and for learning mathematics. During instruction there may have been times when particular methods were mandated by teachers, but generally both in the classroom and in the examinations, students made personal decisions about using CAS or not. The examinations did not contain any questions about CAS use, *per se*, or where there were instructions to use CAS. All students were concerned to understand how CAS outputs were derived; in interviews some students suggested a preference to first perform procedures by-hand when learning mathematics in order to understand how the calculator might be performing algorithms. Following is a discussion of each student's stated preferences for use of by-hand or CAS techniques for learning and assessment.

Ann, Beth and Dana

Ann gave the impression that she was a strong mathematics student, having selected to study two mathematics subjects at Year 12, and appeared to have strong by-hand skills. Ann made pedagogical use of the technology to explore mathematical ideas in both of the mathematics subjects she studied, even though CAS was not allowed in the examinations for the second subject. In the MM(CAS) examinations, she reported that she used CAS for questions involving calculus, factorisation and expansion as its ability to perform routines quickly was very important. Reflecting on her learning, Ann insightfully proposed a link between her by-hand techniques and ability to use CAS:

> "I still learnt everything because we had to know how to do it by hand to use the calculator".

It might be that Ann wanted to practice by-hand skills to know where CAS outputs had come from. It could be that Ann recognised the need to understand mathematics in order to be able to choose a correct procedure on CAS and also to enable a correct CAS input and interpretation of output. For example, Figure 1 shows one multiple choice question and CAS outputs from the three brands in use in 2002. A TI89 user could immediately identify the correct response once CAS was used, whereas users of other CAS would need to recognise answer B as equivalent to the expression provided by their CAS.

If $y=\log_e(\cos(2x))$, then $\frac{dy}{dx}$ is equal to: **A** $2\tan(2x)$ **B** $-2\tan(2x)$ **C** $\frac{1}{\cos(2x)}$ **D** $\frac{-1}{2\sin(2x)}$ **E** $-2\sin(2x)$

CAS OUTPUTS: **CASIO FX2.0** $\frac{-2\cdot\sin(2X)}{\cos(2\cdot X)}$ **TI89** $-2\cdot\tan(2x)$ **HP40G** $\frac{-(2\cdot SIN(2\cdot X))}{COS(2\cdot X)}$

Figure 1. MM(CAS) Exam 1 Q 14(VCAA, 2002) and CAS outputs using inbuilt differentiation capability.

Ann also believed that by-hand skills were required to compensate for times when use of CAS may be time consuming:

> I needed to know how to do it by hand so if the calculator was taking too long to think of an answer I could just go, 'oh yeah', and do it by-hand, and so it was like I was learning it, but then I could just use the calculator. But I still knew it [the mathematics].

An interpretation of this response could be that for certain procedures the time for entry of syntax and for the CAS to apply inbuilt algorithms was greater than the time it would take to perform the solution by-hand or mentally (e.g., equating coefficients when the resulting simultaneous linear equations are easy to solve by inspection). Alternatively, Ann could be referring to the way in which some problems can be solved by a "brute-force" method with a lot of calculation, or by an insightful solution requiring little calculation (the question in Figure 2 provides an example). Ann appreciated the importance of maintaining by-hand skills and her comments suggested that she was good at performing them. Ann also said she enjoyed using CAS, and she demonstrated that she sometimes chose to use CAS on questions within her by-hand repertoire.

Beth's experience of learning with CAS was less positive. She used CAS to perform routines such as factorisation in the examination because it was quicker than by-hand. However, Beth felt that using CAS during the year had resulted in less practice with by-hand skills. She explained that the consequence was that when she experienced difficulty using CAS she was not confident to use by-hand skills instead, due to insufficient practice. Beth felt that she tended to go straight to CAS even though she recognised that for some problems by-hand techniques may be more efficient. Even so, Beth showed us one examination question where she started to use CAS and then stopped herself, recognising that a by-hand technique would be better. This incident also demonstrates how even while solving problems in a high-stakes examination, students make minute-by-minute decisions about efficiency of methods for solving parts of problems.

Dana gave the impression that she had very good by-hand skills and she expressed a clear preference for by-hand work rather than CAS use in class and in examinations. One reason she gave for this preference was the ability to trace errors. The capacity to see intermediate algebraic steps when working by-hand was valued by Dana as it enabled her to look for mathematical errors if she was experiencing difficulty finding an answer. In contrast, she perceived that checking CAS work required checking syntax for entry errors.

> When [solving by-hand] and you come up with the wrong answer [you can] see where you've gone wrong, whereas [for] some CAS functions you get a comma [which is missing] or you get a one or zero and it's all wrong and you don't know where it went wrong.

Dana said that she mainly used CAS to check by-hand work and when she "just can't be bothered" solving by-hand. She gave the example of the quadratic formula as one routine where she would use CAS, although she could perform the by-hand routine if required.

Cate

Cate reported having some difficulties with maths but she was confident with CAS, and stated that she would have enrolled in an easier mathematics stream without it. Cate said that she attempted to use CAS for practically all questions on the examinations, especially to answer questions quickly. When she solved a problem by-hand, she used CAS to check answers. Cate commented that she felt she had lost some by-hand skills, for example the

ability to factorise an expression. She believed that she relied too heavily on CAS but this may be related to the fact that she used it to compensate for her initially weak skills.

Cate indicated in the interview that she was generally able to recognise where the various components of a CAS output came from, based on the by-hand work demonstrated on the board by the teacher. Although Cate was content to use CAS as a "black box" to provide answers, she also wanted to understand how the CAS output was obtained.

> …although we learnt the by-hand work behind techniques…we didn't really apply it [the by-hand techniques]…you could see the outcome which made it [CAS output] make sense…[For example] when we did standard differentiation we'd do it all on the board, the long way [by-hand] …and then eventually you get an answer. But with the calculator you'd put in the equation and get the answer and you can sort of visualise where it came from [due to doing some on the board by-hand]…

Cate commented several times in the interview that she would go back to by-hand work for problems where the CAS output was in a different form to a given answer, such as the answer in a book. She identified algebraic fractions as causing her particular difficulty in this regard, and demonstrated use of the "proper fraction" command in order to alter the algebraic form of expressions involving fractions:

> ...The [type of] answer that threw me out sometimes…[was when] you put it all over, say, four [when entering CAS syntax] and then it might change to a quarter [due to inbuilt simplification algorithms]. We need to know there's a fraction…that sort of thing throws me …the book will have a different answer. The same answer but written differently. So it throws you out sometimes….generally you just use the *proper fraction* [command] and it will rearrange it for you.

This issue of equivalent forms was evident on a day-to-day basis when students were checking answers in the back of their textbook and also when answering multiple choice problems in examinations. Cate commented on both of these, illustrating the unease that occurs when an answer is obtained in an unexpected form (see Figure 1 for an example).

> ...There was one question [on the exam]…using the calculator I didn't have any of the multiple choice responses and it freaked me out….there's nothing worse than having an answer that's not there and then [you] sort of have to stop and really read the equation and see what's there …and then, because it's like the calculator [output] is the answer, whereas it [the form of the multiple choice response] might have been the second or third step [in the automated procedure] before the final answer [given by CAS]….that's when you have to go back to your [by]hand work.

Cate believed that this necessity to find equivalent forms *"kept the mental work and the [by] hand work going"*. Cate's comments suggest that reliance on CAS resulted in less facility with by-hand skills, but she also acknowledged that the need to reconcile given answers with CAS outputs resulted in acquiring by-hand skills. These comments may seem contradictory. Possibly Cate did not consider the finding of equivalent terms by-hand as "by-hand work" as she had used CAS to find the original answer.

Cate's expertise in using CAS for the standard techniques was evident in her preference for multi-step procedures where she could combine a number of steps into one line of working. For example, when finding a stationary point she liked to use a multi-step procedure (differentiate, then solve to find where derivative is zero), entered as one line of syntax and she described this as:

> ...put the solve and the differentiation all in the one hit….

Cate also said that she *"would always reach for the CAS"* for questions requiring exact answers. It could be argued that Cate wasn't demonstrating judicious use of CAS here as many problems involving exact answers are relatively simple using by-hand techniques.

Eve

Eve found use of CAS interesting. She experienced some difficulties with mathematics, and used CAS to overcome lack of facility with by-hand skills. Like Cate, she would have enrolled in an easier mathematics stream if CAS had not been available as support. Her comments about CAS use suggested that she was a very competent CAS user and she enjoyed using CAS as she viewed it as a modern technology. Eve was a very thorough learner, checking her work using multiple approaches as often as possible. Eve generally stressed the usefulness of CAS for checking answers and reported that her teacher had told her to solve problems by-hand and then check using CAS in the exam. She constantly compared her by-hand work and the CAS output. She used both methods to "check" the other method, ultimately trusting CAS more than her by-hand answer.

> …I will trust the calculator more than I will trust myself doing it by-hand.

When Eve was asked whether she used her CAS in the examinations she indicated that she used CAS for most questions. Eve said that when she made the decision to use CAS in the examinations, she felt "lazy". This could be because students knew that they needed to show appropriate working to access marks in the examination. She suggested that CAS use in examinations was high for most students in her school. This contrasts with the findings about use of graphics calculators in a calculus course by Boers and Jones (1994) who found that students under-utilised the technology. They suggested that students were preferencing algebraic approaches over graphical approaches. It would be interesting to consider whether students are using CAS more than teachers might expect and whether ability to use algebraic features of CAS may contribute to the high level of CAS use.

> As soon as the exam started everyone was …[here Eve demonstrates by starting to tap vigorously on her calculator keyboard to make a noise]. You could hear [tapping]. All you could hear the whole time [was tapping on keyboards].

Eve demonstrated expert CAS use on several questions, where she valued the speed. For example, she adeptly solved a trigonometric equation in a restricted domain using CAS This demonstrated good use of the history capability of CAS. Eve's teacher had shown them how to solve equations of this type (solving, and restricting the domain) using nested procedures and modifying the entry line, as shown in Figure 2. This example also illustrates that many examination questions which can be solved insightfully with very little calculation (such as this one) are generally tackled with brute-force calculation in the time-pressured environment of an examination, whether technology is available or not.

> …I'm doing "solve sin(2x)=1". Solve for x given that x>0 and then you go "AND" x<4π. I got all the [solutions] $x=5\pi/4$ etc and then, because [the question asks for the sum of solutions], I've got to bring it back into the little line screen [entry line], go through and replace the "ors" with the plus sign [+] and get rid of the $x=$…You could do this by-hand, but it would take a lot longer...you plus them all together and you get 7π.

Eve felt that in some ways having a CAS meant that she didn't have any written mathematics notes, as inbuilt CAS algorithms were the equivalent of notes which would normally be recorded in a book.

> …you go to the start of the chapter and they've got all the little green boxes in [the] textbook, where it's got remember this [routine procedure], remember that [routine procedure]. You don't remember things like that [when you have CAS] because they're done on the calculator...

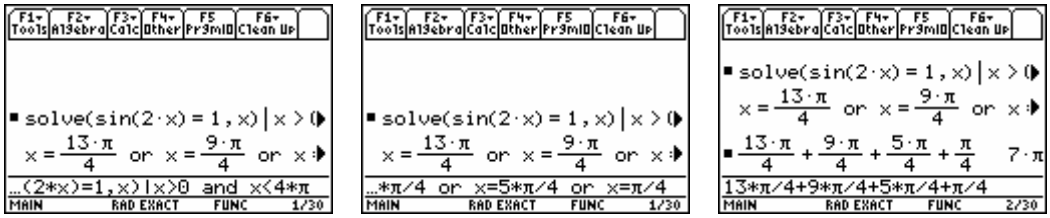

Figure 2. Finding the sum of solutions of the equation $\sin(2x)=1$ for $0 < x < 4\pi$.

Even though Eve believed that CAS was the equivalent to her notes and recognised that there were inbuilt algorithms in CAS, she had not recognised the extent of the algorithms built into CAS. For example, she appeared not to understand that CAS could differentiate a product. Eve believed that in order to find the derivative of a product of two functions, she would have to assemble partial answers from the CAS into the product formula herself:

> Because you can get it [the derivative] on your calculator and you can't do that [the product rule]. You have to do that by-hand. Like you can do little parts [of the product rule], you can do the derivatives [of each component function]……You can do the *d/dx* on the calculator. ..you just have to set it out [the product rule] by hand and then you know, bracket it off, put it in the calculator….

For example, to find the derivative of the product $\sin(x)x^2$, Eve would write down the product rule first. She would then work out the derivatives of $\sin(x)$ and of x^2, possibly using CAS, and substitute these by-hand into the product rule. Eve's reason for not just entering the entire expression into the CAS and using CAS to differentiate in one step is that she didn't think that the CAS would *know* that the product rule was required. Eve stated that she generally did questions involving the product, chain or quotient rule for differentiation by-hand and then went back and checked her results using CAS. Her confusion on this point is probably connected with her concern, shared with her teacher, that CAS sometimes gave answers in unexpected forms. The teacher's suggestion of starting with by-hand work generally provided exactly one of the answers to the multiple choice questions on the examination. These answers were constructed by people familiar with the by-hand methods institutionalised in the local area. This privileges particular forms of algebraic expressions, which may differ from those privileged by the algorithms built into a CAS. Identifying the correct multiple choice answer may be time-consuming.

Learning To Use CAS

Students were asked what advice they would give to a friend who was going to study MM (CAS). This simple question provided insight into what students identified as critical factors for achieving success. There seemed to be two aspects to this advice: what was required to be a good CAS user (students knew that researchers were interested in CAS) and also what was useful to maximise mathematical learning and hopefully their very important examination results.

Ann valued being able to make sensible choices about efficiency of CAS or by-hand techniques and this comment suggested a focus on performing well in an examination. Ann stressed the importance of knowing both by-hand and CAS techniques efficiently:

> Learn how to use the calculators so that you're not stuck and just don't assume that the calculators will do everything. Know how to do it by hand as well so you can decide which is quicker…

Ann commented that she liked to use CAS as it gave her the "…*opportunity to do something different and to explore*…" stating that she really enjoyed studying the CAS

subject. This statement suggests that Ann used CAS in pedagogical ways, as well as functional ways.

The focus of Beth's advice to a friend was to learn to use the CAS commands. Beth stressed the importance of listening carefully in class to CAS instructions. She tended to write down both CAS instructions and descriptions of what the CAS could be used for.

> I tended to write…the button names and stuff that … a brief summary of what it actually did. Or just say you need to differentiate [so] go to CAS and it will differentiate it.

Cate stressed the importance of knowing features of the CAS and how to use them. She said that she normally learned how to use CAS from the teacher but she also learned CAS from other students and often sought their help. Cate reported many discussions outside of class time about what buttons needed to be pressed to access particular CAS features.

> .mainly with processes…which button to press or where say [particular calculator button was]…how you had to go through catalogue to get through to it [the CAS feature].

Interviewees also confirmed that peer networks, within and across classes, were important for sharing technical information about using CAS. Comments suggested that the teachers ensured that students shared new CAS discoveries with their classes. Only one student (Ann) said that she read the manual to learn about CAS and she commented that the amount of extraneous material in the manual caused difficulties. Ann also learned to use CAS through personal exploration, and from her peers and her teacher. Ann suggested that her teacher was quite enthusiastic about sharing discoveries about new CAS features, showing these to the class using an overhead projector. Her teacher also advised students on programs available to download onto calculators. Beth said one student in her class found downloadable programs for the CAS and this student became the CAS expert in the class, troubleshooting CAS queries. Eve said that she learned to use CAS through the teacher demonstrating procedures using the overhead projector and she also valued the CAS advice provided by the research team in a booklet provided to students and teachers. Faure and Goarin (cited in Trouche, 2005) in a study of French year 11 students' relationship to calculators found that most learning about calculators occurred outside the classroom rather than through teacher instruction. This appears to be a different situation to that described by the five students interviewed who stressed the important roles played by teachers in learning about CAS in addition to the support provided by peers. Teachers were identified as important in the learning of CAS techniques in these classrooms.

Conclusion

The interviews showed how the students used CAS in the examinations and their experiences of learning with it. The five students had different strengths, difficulties and preferences, which indicate some of the range of reactions which teachers must deal with. The students also demonstrated some commonalities. In this setting where the importance of high marks on the examinations consisting mainly of predicable questions dominated both teachers' and students' approaches, the main reason for CAS use was getting answers quickly, although some used it because they did not trust their own by-hand skills. Checking answers was another major use. In at least one instance, it was clear that the student had not appreciated the power of CAS. Students reported that they actively considered whether to use CAS or not, question by question. In this circumstance, we value students being discriminating users of CAS so that they do not use it without considering whether there are more insightful or efficient methods available. Pierce and Stacey (2004b)

stress the importance of this development of judicious use of CAS. The interview data did not enable us to make judgements about the extent to which students were discriminating users, but it highlighted that any judgement of being a discriminating user has to be made with knowledge of the variation in the strength of students' by-hand skills. Students with weak by-hand skills must make different judgements than students with strong skills. In contrast to studies in other countries (e.g. Trouche, 2005), students reported that their teachers played an active role in helping them master the technology.

Students considered issues associated with use of CAS in a very reflective way. They had thought through many of the issues that we might have anticipated that only teachers deal with. Access to CAS did not mean that students were happy to ignore by-hand skills or neglect understanding intermediate steps: indeed it could be argued that by-hand skills were uppermost in students' minds. Some knew, however, that they could manage with less technical proficiency. All of the students were adamant that use of CAS required what we call algebraic insight (Pierce and Stacey, 2004a) and several commented especially on the need to be able to move fluently between different forms of algebraic expressions; a key component of algebraic insight.

A very positive comment is that two students felt confident to attempt Mathematical Methods (CAS) at year 12 where algebraic skills may have otherwise prevented them, and their responses related to this subject were positive. Taken over all, three students were very positive about their experiences in MM(CAS), one (Beth) was negative and another (Dana) was positive although she preferred not to use the machine. Their different reactions reflect some of the diversity that exists in senior mathematics classrooms.

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