Algebraic Thinking in the Numeracy Project: Year One of a Three-Year Study

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In this first year of a three-year study, we report on the abilities of groups of students from years 7-10 to generalise numerical operational strategies involving compensation adjustments in the four arithmetic operations. The Year 7 students, who had all participated in the Numeracy Project, were more successful than those in other year groups in using literal symbols to express generalities. The Year 9 students, about half of whom with Numeracy Project experience, were the least successful despite exposure to formal algebra.

A key element of the benefits experienced by all students who have participated in the Numeracy Project, (New Zealand Numeracy Professional Development Projects, 2004) has been the growth of their number sense (see Bobis, et al, 2005. pp. 43-57; McIntosh et al, 1997, p. 3). Improvements have occurred in students' number knowledge and in their ability to respond to numerical situations by applying, sensibly and flexibly, a range of mental operational strategies.

A class of such strategies involves a range of compensation actions. For example, 34 + 19 can be transformed to an equivalent 33 + 20 by first increasing the 19 by 1 then making a compensatory adjustment by reducing the 34 by 1. This task illustrates compensation in addition. Similar compensation actions can be applied where appropriate to subtraction, multiplication and division tasks. Students who apply these and other operational strategies to sensibly solve problems show an awareness of the relationship of the numbers involved in the problem. And in our view they demonstrate that the strategy is generalisable and so are engaging in algebraic thinking. This connection between an awareness of generality in any mathematical domain and algebraic thinking is well supported by the views of Fujii (2003), Fujii and Stephens (2001), Kaput and Blanton (2001), Lee (2001), Mason (1996), and Steffe (2001).

Fujii (2003) and Fujii and Stephens (2001) extended this link between number and algebraic thinking by arguing further that, within the strategies that students devise as above and in which generality of thinking is illustrated, the numbers themselves act as variables. They refer to these numbers as quasi-variables, which Fujii (2003) elaborates as:

> a number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used are. (p. 59)

A corollary to the proposition that these numerical operational strategies are algebraic in nature, is that here we see algebra in arithmetic not algebra as a transition from arithmetic that is typically the case in the early secondary years of school mathematics.

This work, related to using quasi-variables, was central to a study that we carried out in 2003 to examine whether students in the numeracy project were more successful than comparable children not in the project in applying sensibly a range of operational strategies in addition, subtraction, multiplication and division. The results of the study indicated that they were (Irwin & Britt, in press). In 2004, we extended the earlier work to see if different groups of students could demonstrate this algebraic thinking with decimals as well as with whole numbers. They could (Irwin & Britt, 2004).

We have now embarked on an investigation to compare the extent to which students, who have participated in the Numeracy Project at Intermediate School and those who have not, have extended their algebraic thinking from quasi-variables to using algebraic variables as they progress from Year 8 (age 12) through 10 (age 14). For example, the compensation in addition task illustrated earlier can be generalised as $a + b = (a - c) + (b + c)$. In our view, students who show understanding of the structure of earlier compensation tasks in addition, and can also complete $a + b = (a - \square) + (b + c)$ successfully will have demonstrated understanding of the role of algebraic variables in representing the generality of the structure of the compensation in addition strategy. We expect that Numeracy Project students will perform better on such tasks than those who have not participated in the project. We are nevertheless aware that given that secondary school algebra teaching does not at present build on students' understanding of using numbers as quasi-variables, it may be that an interference mechanism (Pesek & Kirshner, 2000) could limit the growth in expressing generality from quasi-variables to full algebraic variables. We intend to explore this and other related questions as we follow individual student's pattern of achievement as they respond to the same test items over the next three years.

As 2004 was the first of these three years, we cannot at this stage make any longitudinal statements about students in the numeracy project. However, we can compare year groups by looking at students who have come from schools where the Numeracy Project was in use with those from schools that were not involved in the project. We can also look at patterns of achievement across age groups.

Method

Participants

Students came from four intermediate schools, two in Wellington and two in Auckland, and their neighbouring secondary school. While all intermediate schools had participated in the Numeracy Project none of the four secondary schools had. The schools were chosen because of the relatively close match of the decile ranking (socio-economic ranking of the school's community) of the intermediate and secondary schools that most students are likely to attend. The ethnic composition of students at these schools is shown in Table 1.

Table 1
Characteristics of Schools in the Three Year Study of Algebraic Thinking

School Type	Decile ranking	Student roll	Asian	Maori	N.Z. / European	Other	Pasifika	Date of ethnicity data
Intermediate	2	216	3%	30%	46%	-	21%	11.04
	3	528	8%	28%	28%	12%	24%	11.03
	5	628	-	17%	66%	15%	2%	6.02
	6	330	3%	15%	73%	5%	4%	5.03
Secondary	3	795	4%	29%	58%	-	9%	7.02
	4	1435	11%	23%	45%	-	21%	5.04
	5	1493	6%	14%	71%	6%	3%	11.04
	7*	1253	3%	18%	73%	2%	4%	8.02

*no tests given in 2004

For reasons that suited the schools, three intermediate schools gave the test to all students in three or four classes, usually selected by the willingness of the teacher to participate. The fourth school taught mixed classes of Year 7 and Year 8 so decided to give the test to all students. Altogether, 98 Year 7 students were assessed. One secondary school chose not to participate in 2004 but is expected to be involved in 2005 and 2006. Three secondary schools gave the test to their Year 9 students and two schools gave it to their Year 10 students. Details of the sample are presented in Table 2.

Table 2
Number of Schools and Students Participating in Year 1 of the Study

Year	Number of Schools	Decile of the school	Number of students participating
7	1	2	98
8	4	2, 3, 5, 6	317
9	3	3, 4, 6	781
10	2	4, 6	549

Materials

A test of algebraic thinking with four sections: addition, multiplication, subtraction and division, was devised by the first author and trialled with Year 8 students in a decile 10 intermediate school and with Year 9 and 10 students in two secondary schools (one decile 3 and the other decile 10). Each section included two exemplar models showing compensation adjustments and five items requiring students to make adjustments similar to those in the models but with an increasing level of understanding of number and of algebraic generality. For example, in Section A involving compensation in addition, students were to use *Jason's method*, illustrated by 27 + 15 being transformed to 30 + 12 giving a total of 42 and also 34 + 19 being transformed to 33 + 20 for a total of 53. The five students' items were in order: 298 + 57, 35.7 + 9.8, $58 + n = 60 + \square$, $9.9 + k = 10 + \square$, and $a + b = (a + c) + \square$. The five items in each of the other sections were similar to these items in that they progressed from first using whole numbers as quasi-variables, then decimal fractions as quasi-variables before generalising with a mix of whole and decimal numbers as well as literal symbols. The final item in each section required students to complete an algebraic identity with literal symbols only.

Procedure

The teachers administered the test towards the end of the fourth school term in normal class time on a day that suited them. Students were instructed to read the shaded section with the two exemplars carefully, write the answer in the space below each question, and to not use a calculator. Graduate students, who had just completed their pre-service secondary mathematics teacher education programmes, marked the tests under the guidance of the authors. Responses were credited as correct if they followed the structure of the exemplars.

Results

Tables 3 through 5 and Figures 1 through 5 show the percentage of students in each year group that solved each item in the manner illustrated by the exemplars.

We were surprised at the marked difference between year groups in the percentage of students with some literal items correct. Graphs for different year groups are given in Figure 1.

Table 3
Overall Percentage of Items Completed Accurately by Year Group

Year	Mean score	Modal score	% of students with some literal items correct
7	4.95	1	46%
8	5.2	3	26%
9	3.9	0	18%
10	5.7	0	31%

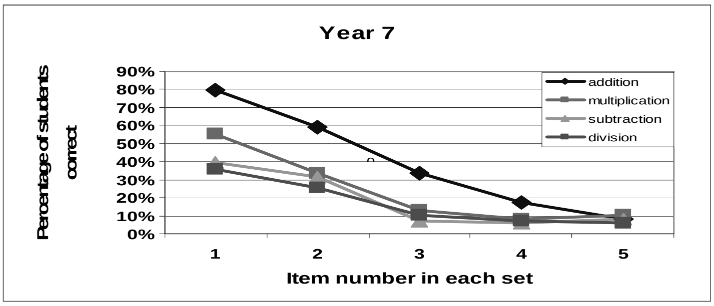

Figure 1. Results of Year 7 students on the test of algebraic thinking (1 school, 98 students).

A higher percentage of Year 7 students were successful on these items than were other year groups, an issue that will be discussed later. The pattern of increasing difficulty within each page and operation that this year group demonstrated is the same for all year groups. For Year 7, unlike other year groups, division was the most difficult operation.

The graph of the percentage of Year 8 students who succeeded on these items (see Figure 2) showed a much sharper decline between item 2 and item 3 in addition than did the graph for year 7 students. It also showed subtraction to be the operation on which fewest students succeeded.

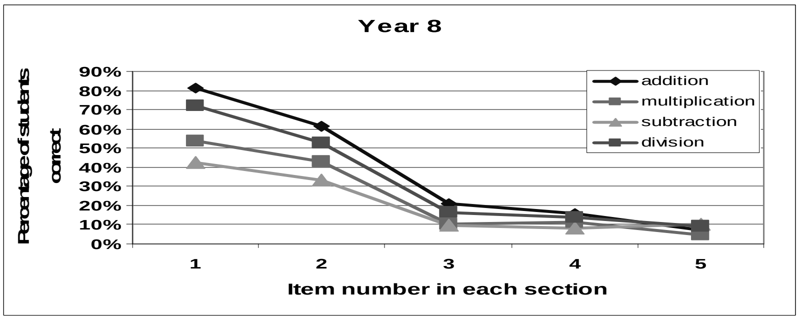

Figure 2. Results of Year 8 students on the test of algebraic thinking (4 schools, 317 students).

Table 4 compares results of the four intermediate schools. As schools chose which students to include these data may not represent the whole school, except for the decile 2 school.

The differences between year 8 groups will not be important in future years as each child will be compared against their own score in later years. There is nevertheless some interest in the fact that the school with the lowest decile ranking, and which did not select students, had a higher average than any of the other schools in the percentage of students who were correct on some items that included literal symbols. They appear to have done a better job at this than did the selected classes from other schools. Also, the school with the highest decile ranking had higher modal scores but few students who transferred this understanding of using numbers as quasi-variables to the use of literal symbols.

Table 4
Scores of Year 8 Students from Four Intermediate Schools

Decile ranking	Number of students	Mean score	Modal score	% of students with some literal items correct
2	82	4.96	1	46%
3	66	4.68	0	37%
5	76	5.66	0	38%
6	93	5.61	3 and 6	19%

Year 9 students were, on average, less successful than any other year group (see Figure 3). We believe that this may relate to lack of continuity in working with numerical operational strategies and therefore quasi-variables between intermediate and secondary school. It may also be that somehow the secondary teaching approaches created a cognitive interference among the students who had previously had exposure to Numeracy Project approaches.

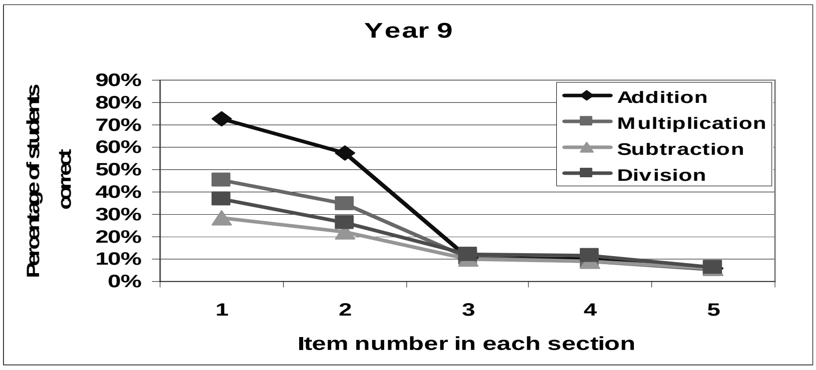

Figure 3. Results of Year 9 students on the test of Algebraic thinking (3 schools, 781 students).

When we compared the scores of Year 9 students who had attended intermediate schools that were using the Numeracy Project (NP) with those who had not (NNP), we found no appreciable difference. Both groups had a mean score of 4.3 (NP, 4.31 and NNP, 4.28). Both groups found addition to be the easiest operation and subtraction to be the most difficult. The students from Numeracy Project schools performed slightly better than those from non-Numeracy Project schools on the first item in each section (see Table 5).

Year 10 students did somewhat better than Year 9 students on this assessment but the pattern of achievement was like that for other year groups (see Figure 4).

Our particular interest was in students' ability to generalise algebraic thinking from numerical items, something that they may have learned in the Numeracy Project, and also their ability to express this algebraic thinking with letters as variables. Therefore we analysed the students who were successful on some numerical items as well as on some literal items (see Figure 5). These results also appear in Table 3.

Table 5
Percentage of Year 9 Students Correct on Initial Item for the Four Operations by Numeracy Project Schools and non-Numeracy Project schools

Numeracy Project participation	Number of students	Addition	Multiplication	Subtraction	Division
From Numeracy Project Intermediates	402	75%	45%	29%	36%
From non-Numeracy Project Intermediates	310	70%	45%	27%	35%

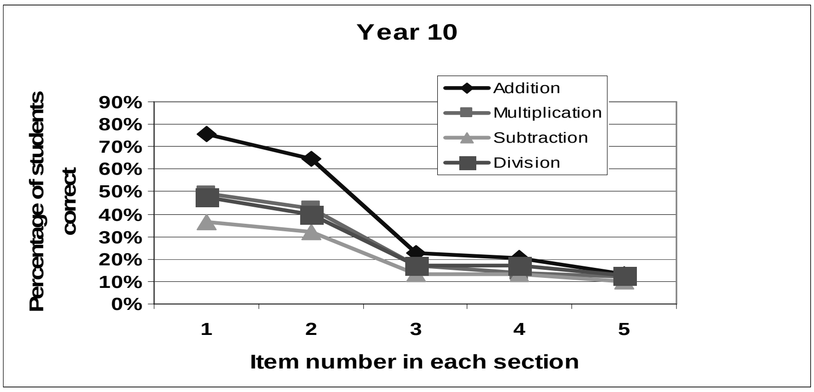

Figure 4. Results of Year 10 students on the test of Algebraic thinking (2 schools, 549 students).

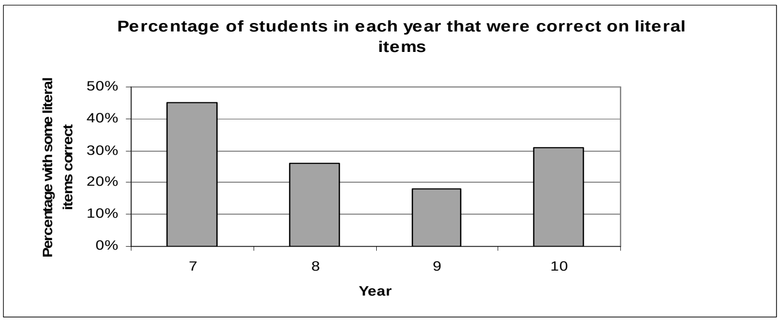

Figure 5. Percentage of students from all years who were correct on both numerical and literal items.

Surprisingly, Figure 5 shows the superiority of the Year 7 students in progressing from generalising the strategies using quasi-variables to generalising with variables. The Year 9 students were least successful with algebraic generalising even though they were currently being taught algebra in a conventional manner. The data in this figure provides an excellent base line for determining if the new Secondary Numeracy Project will help students build new algebraic skills on their existing ones.

Discussion

In this discussion we focus on two particular aspects. The first relates to the students from all classes who appear to be in transition, that is, thinking algebraically on numerical items and beginning to transfer this algebraic thinking to literal items. In the second, we discuss possible reasons for the superiority of the Year 7 group.

Students could complete up to 8 of 20 items correctly if they used algebraic thinking with numerals only. This happened in the decile 6 intermediate school, whereas in the decile 5 intermediate school only one student who scored 8 or less was correct on at least one literal item. In the decile 2 school, 19 students who scored 8 or less had some literal items correct and in the decile 5 school 15 students who scored 8 or less had some literal items correct. These students were actively engaged in algebraic thinking. We do not know exactly what teaching had occurred in their classes to encourage this thinking, but it would be worth exploring and fostering. Similarly it would be useful to explore the teaching in schools where students were able to use numerals as quasi-variables but could not transfer this thinking to the use of letters. Most of the students who were accurate on some literal items scored at least a total of 5. We therefore nominated students who scored a total from 5 to 15 as being transitional in the development of algebraic thinking. Those scoring from 16 to 20 were classified as experts. Eighteen Year 8 students scored in this expert range.

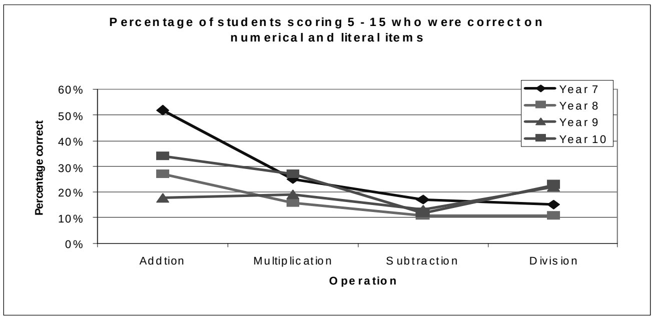

Figure 6. Percentage of students from all years who were correct on both numerical and literal items.

Figure 6 shows that addition was the easiest operation for transferring algebraic thinking from numbers as quasi-variables to letters as variables, in years 7, 8 and 10, but not for Year 9 where success on literal items was generally low. It also suggests that work of this kind with subtraction may be more difficult that many of us have believed. And overall it raises the question of whether or not year 8 students in general, spend time on the use of quasi-variables in addition at the expense of the other operations or just whether more time needs to be devoted to exploring operational strategies involving subtraction, multiplication and division. It again demonstrates the superiority of Year 7 and Year 10 students on this assessment of algebraic thinking.

Understanding letters as representing general numbers, rather than as specific unknowns, has long been difficult in secondary school algebra (Küchemann, 1981). These results suggest that students whom we have designated transitional are ready to extend their work from using numbers as quasi-variables to expressing algebraic relationships using letters for general numbers. The 18 experts found in this sample of year 8 students are already comfortable with this use of letters.

Why were the Year 7 students more successful than the other year groups? Again we can only speculate. The fact that they were better than the Year 8 students in their own school is another intriguing question. The Numeracy Project facilitator for this school reports that when working with the teachers in this school he used the term variable, representing it first with an empty square and then with a letter. He also held a workshop in which he used examples of algebraic thinking taken from seminar presentations by the first author. It seems possible therefore that the teachers may have introduced the concept of variable in their classes. It may also be that the Year 7 students were more successful than the Year 8 students in their own school because the older students may have been introduced to algebra in a traditional manner that did not grow out of algebraic thinking with numbers as quasi-variables. We will watch this cohort in future years with particular interest, and as well take note of the teaching that they receive as Year 8 students.

Two of the secondary schools in this study are now also involved in the Secondary Numeracy Project. This involvement will enable us to see if that project is able to avoid some of reduction in algebraic thinking noted in this year's cohort of year 9 students. It will be an intriguing on-going study.

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