Conceptual Understanding of Spatial Measurement[1]

Margaret Curry
Macquarie University
<curryclan@ozemail.com.au>

Lynne Outhred
Macquarie University
<lynne.outhred@mq.edu.au>

Although there has been considerable research into students' learning of length, area, and volume measurement, there is very little research into their relationship. This paper describes a developmental study of how 96 students in Grades 1-4, randomly selected from six public schools across the Sydney metropolitan area, related measurement in these three domains. Two methods of measuring volume (by filling and by packing) were included. The results indicate a clear increase in students' understanding of unit structure of length, area and volume across Grades 1 to 4 with volume (packing) being the most difficult. Volume (filling) was closely aligned to length. These results support the wisdom of highlighting the similarities and differences in the unit structures of length, area, and volume measurement in a teaching context.

Apart from general surveys such as NAEP and TIMSS, we know of no measurement studies which have simultaneously investigated students' understanding of length, area, and volume or the relationship between them. Several studies have compared two attributes, however. Some studies have addressed the confusion between perimeter and area (e.g., Lehrer, Jenkins & Osana, 1998); Outhred and Mitchelmore (2000) highlighted the importance to area measurement of a conceptual understanding of length measurement; and Battista (2003) discussed the relation between area and volume measurement. Yet there are good reasons to expect students' understanding of length, area, and volume measurement all to be related, and there are many advantages to be gained from knowing about their relationship.

Relation between length, area, and volume measurement

Length, area, and volume measurement are theoretically related in several ways. To simplify matters, we shall limit the discussion to rectangular shapes.

- Length, area, and volume are all spatial domains. Fundamental to all of them is the conservation of the corresponding attribute.
- Length, area, and volume measurement have many common features shared by all measurement activities, such as the importance of unit iteration and the relation between measure and unit size.
- The measurement of each attribute leads to a definite *unit structure* (the pattern formed when the units fill the object to be measured), and the structures are closely related. For length, the unit is iterated in one dimension. For area, the unit is iterated in two dimensions to create a rectangular array. (Internal) volume may be measured in two ways. In one method, the space is packed with a three-dimensional array consisting of a two-dimensional array of units which is iterated in the third dimension. In the second method, the space is filled by iterating a fluid unit which takes the shape of the

[1] The study reported in this paper was funded by the Australian Research Council under their Project-Linkages grants scheme. The Industry Partner is the New South Wales Department of Education and Training (DET), whose support and cooperation is deeply appreciated. The authors also wish to thank Mike Mitchelmore for his input into the research and the writing of this paper.

container. In this method, the unit structure is one-dimensional. To differentiate these two methods, we shall call them *volume (packing)* and *volume (filling)* respectively.

- The measurement of area and volume leads to multiplicative relationships involving the lengths of the sides.

Furthermore, several empirical results suggest that the order in which students learn about measurement may be similar for the three attributes:

- The early errors of overlapping units, leaving gaps, or using non-congruent units have been reported for the measurement of length (Bragg & Outhred, 2001) and area (Outhred & Mitchelmore, 2000).
- The difficulty of making the transition from filling a space with concrete units to visualising and using the unit structure has been reported for length (Bragg & Outhred, 2001), area (Battista, Clements, Arnoff, Battista & Borrow, 1998), and volume (packing) (Battista & Clements, 1996).
- The above studies are among many which indicate that, for all three attributes, students learn to measure by filling a space with multiple copies of the same unit before being able to measure by iterating a single unit.

Significance of the relationship

If teachers knew more about the growth of students' conceptual understanding of length, area, and volume measurement, they would be better able to teach these topics. And if they better understood how students relate measurement in the three domains, they would be better able to relate the teaching of each attribute to students' understanding of the others.

It was such considerations that led to the *Count Me Into Measurement* (CMIM) program in New South Wales (Outhred, Mitchelmore, McPhail & Gould, 2003). CMIM combines curriculum and professional development by supplying teachers with research findings and coordinating their application in the classroom. The basis of CMIM is known as the *Learning Framework for Measurement* (LFM), initially constructed from the available research evidence and since modified following field trials. The first three levels of the LFM, intended for Grades K-2/3 and covering length, area and volume, are briefly described in Table 1. Formal units and their application are predominantly dealt with in the later levels of the LFM.

However, although there are good theoretical and empirical reasons to expect understanding of length measurement to develop before area, and area before volume, there is simply not enough research evidence to enable the teaching of the various LFM levels in each attribute to be assigned to grade levels with any confidence. It has also proved difficult to advise teachers on how to link the three attributes.

The LFM does not clearly distinguish between volume (packing) and volume (filling). Since the measurement of the volume of rectangular objects by filling is one-dimensional and closely resembles length measurement, we predict that volume (filling) develops in a similar manner to length and certainly before volume (packing). Again, there is no empirical research to confirm this prediction. We know of only one relevant study: Reece and Kamii (2001) explored how students in Grades 1 to 5 iterated a single cup of rice to compare two volumes by filling. They found that the percentage of students who spontaneously constructed equal units increased from 15% at Grade 2 to 66% at Grade 5.

Table 1
The First Three Levels of the Learning Framework for Measurement

Level	Description
1	*Identification of the attribute* includes conserving, directly comparing, and ordering quantities.
2	*Informal measurement* includes choosing and using appropriate units for measuring quantities, and comparing and ordering quantities by using identical units to cover, fill or pack objects without gaps or overlaps.
3	*Structure of repeated units* includes using one unit or a composite unit to work out how many will be needed altogether when making indirect comparisons, and explaining the relationship between unit size and the number of units required to measure an object.

The present study

A study was therefore designed to provide more information on young students' understanding of length, area, and volume measurement and the relationship between them. Specifically, the following research questions are being investigated:

1. Do students' understanding of length, area, and volume measurement show a clear developmental pattern?
2. How are students' understandings of length, area, and volume measurement related to each other?

Both volume (filling) and volume (packing) are included.

This paper reports some preliminary insights obtained from an analysis of part of the data collected. Further results will be presented at the conference.

Method

The research instrument

A 45-minute clinical interview was designed and pilot-tested in 2003-2004 and administered in Term 4, 2004, to a sample of 96 students chosen from six schools that were representative of the variety of public schools in Sydney. There were 24 students in each of Grades 1-4.

The interview consisted of a number of tasks designed to test students' understanding of the measurement of length, area, volume (packing) and volume (filling). In order to allow a valid comparison of development across the four attributes, every effort was made to construct parallel tasks. Pilot-testing showed that parallelism could be satisfactorily achieved for length, area and volume (packing). Because of the fluid nature of the unit, only partial parallelism was feasible for volume (filling).

The items to be reported below were parallel tasks investigating students' understanding of unit iteration across all four attributes and therefore related to Level 3 of the LFM (see Table 1). In each task, students were given an object to be measured and a single measurement unit. (All units gave integral measures.) The four tasks are shown in Table 2. It may be noted that the length, area and volume (packing) questions were almost identical. To ensure no ambiguity occurred, the interviewer clarified the meaning of "the space along" or "the space inside" by moving her hand along the length of the ribbon, over the rectangular region, or within the rectangular box. Neutral probes were used whenever the interviewer was uncertain as to the student's intent or meaning.

Table 2
Tasks used for Assessing Student Understanding of Measurement of the Four Attributes

Attribute	Question asked
Length	*How many rope lengths this size* [indicate] *do you think would be needed to measure the space along this line? Can you show me how you would use this rope to check your answer?*
Area	*How many tiles this size* [indicate] *do you think would be needed to measure the space inside this rectangle? Can you show me how you would use this tile to check your answer?*
Volume (packing)	*How many blocks this size* [indicate] *do you think would be needed to measure the space inside this box? Can you show me how you would use this block to check your answer?*
Volume (filling)	[In clear view of the student the interviewer fills a cup with rice, pours the contents into a jug, and shakes the jug to level the rice.] *The jug now contains one cupful of rice; how many cupfuls of rice would be needed to fill the jug completely? Can you show me how to use this cup to check your answer?*

Data analysis

All student responses were video-recorded and transcribed. Initial categories defining emerging data, and linked to the LFM, were coded and continually refined as further data emerged. Analysis proceeded concurrently with the categorisation, and at the same time searches were made for the same phenomenon appearing within the different attributes of length, area, volume (packing), and volume (filling). The continual collection of data, reformulation of categories, and linkage to the existing literature created a "logical chain of evidence" (Hubermann & Miles, 1998, p. 187) to support the conclusions that were drawn in response to the stated research questions.

As a result of the above process, two scores were obtained for each attribute. The first score (called *estimation*) was obtained from the initial response:

- 2 Provides an estimate, together with an explanation (or a clearly visible action) that demonstrates an understanding of unit structure.
- 1 Provides an estimate, but understanding of unit structure is not evident.
- 0 Does not provide an estimate, or admits guessing.

The second score (called *measurement*) was obtained from the response students gave when they were asked how they would check their answers:

- 3 Correctly iterates the given unit without prompting, either by repeatedly copying the unit to fill the space or by marking units on each dimension of the given object and using a multiplicative process.
- 2 Iterates the given unit so imprecisely that the unit structure is incorrect; but, after prompting, acknowledges the need for a mechanism to maintain the unit size.

1 Iterates the given unit imprecisely and considers this adequate.
0 Does not iterate the unit (correctly or imprecisely).

To simplify comparisons, these scores were treated as if they formed interval scales so that mean scores could be calculated.

Results and Discussion

Figures 1 and 2 show mean estimation and measurement scores for the four attributes. The results demonstrate a clear increase in students' understanding of unit structure from Grade 1 to Grade 4 for all attributes. We shall discuss each attribute separately.

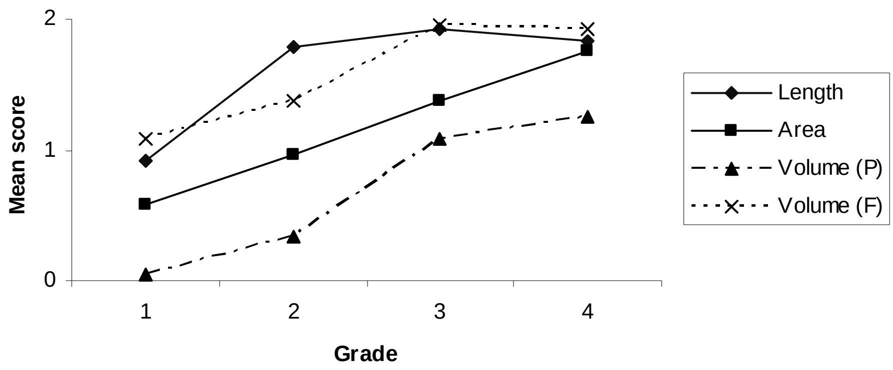

Figure 1. Mean estimation scores for each attribute, by grade.

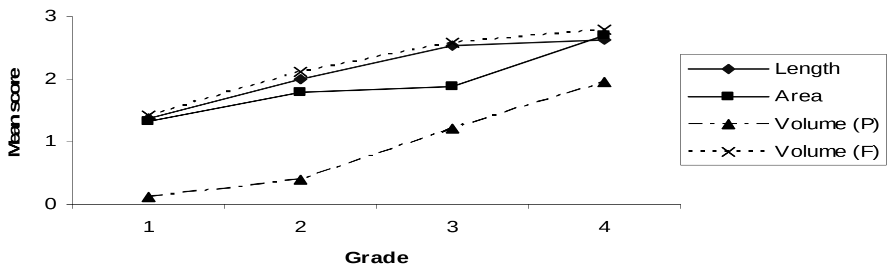

Figure 2. Mean measurement scores for each attribute, by grade.

Length

About half the students in Grade 1 provided responses which indicated some awareness of the unit structure. Both estimation and measurement scores then showed a marked increase to Grade 3, by which time the majority of students seemed to understand the unit iteration structure. There was quite a sudden increase in estimation scores between Grades 1 and 2. This could have arisen because the Grade 1 students were unused to estimating; alternatively, they may not have internalised the measurement process sufficiently well to make estimates confidently. Several Grade 1 students were unable to estimate the length of the line by eye but could measure it when able to manipulate the unit.

Volume (Filling)

Students' scores for volume (filling) closely followed the trend for length, especially for measurement. Most students approached the task as they had done for the length task, treating the height of the rice in the container as if it were a unit length and iterating it, either mentally or using their fingers, up the side of the container. However, the volume (filling) task also required students to fill the cups in equal measure in order to obtain a uniform unit (even in Grade 1, many students carefully skimmed rice off the top of the cup to ensure that the amount remained constant), whereas no such procedure was necessary for length measurement. The left-hand side of Table 3 illustrates the association between students' measurement scores for length and volume (filling); a similar association was found for the estimation scores. It will be seen that several students found length easier than volume (filling) and vice versa, suggesting that there was no direct link between students' understanding of the measurement of the two attributes.

The results thus confirm the researchers' expectations that, wherever comparison is possible, volume (filling) would show a similar development to length. But they also show that the measurement of the two attributes actually involves slightly different mechanisms.

Table 3
Relation of Length Measurement Scores to Volume (filling) and Area Measurement Scores

Length score	Volume (filling) score				Area score[a]			
	0	1	2	3	0	1	2	3
0	4	5	1	1	7	2	0	2
1	0	6	1	11	3	9	2	4
2	0	3	3	8	1	4	2	7
3	0	9	11	33	6	7	3	36

[a]N = 95 because of one missing data point.

Area

Area estimation and measurement scores increased steadily from Grade 1 to Grade 4. In Grade 1, students' estimation scores were lower for area than length, but the measurement scores were approximately equal. Most students were at least able to move the unit around the rectangle in a rough covering procedure (thus scoring 1 for measurement), but several had not sufficiently internalised this process to be able to make estimates confidently. By Grade 4 most students could estimate and measure area as well as they could length.

Students who were successful on the area measurement task generally iterated the edges of the tile along the sides of the rectangle, making repeated marks, in a similar fashion to the way they approached the length measurement task. This observation would suggest that length measurement was a prerequisite for area measurement. However, as the right-hand side of Table 3 shows, the relation between students' performance on the length and area measurement tasks was not so simple: There were several students whose area scores were higher than their length scores. Closer analysis of student responses suggested that both length and area measurement scores were affected by how carefully the student placed the iterated unit and recorded its successive positions.

Volume (Packing)

Most students in Grades 1 and 2 had little or no idea of how to estimate or measure volume (packing). Some students only covered the base of the given box; others said they did not know what to do, or needed more cubes. Between Grade 2 and Grade 4, there was a substantial increase in both estimation and measurement scores, but Grade 4 scores were still well below those for area. Three factors seemed to make this task more difficult than the area task: (1) Students had to move the unit around in empty space instead of on a hard surface; (2) they could not record successive positions of the unit as they moved it around; and (3) it was more difficult to find the larger number of units. Only the older students successfully completed this task; they seemed to have a mental picture of the unit structure, often finding the number of units in the base before multiplying by the number of layers.

Table 4
Relation Between Measurement Scores for Area and Volume (packing)

Area score	Volume (packing) score			
	0	1	2	3
0	16	0	0	1
1	17	2	1	1
2	5	0	1	1
3	20	5	4	20

Note. N = 94 due to two missing data points.

Table 4 shows the relation between students' measurement scores for area and volume (packing). It strongly suggests that an understanding of area measurement is a prerequisite for being able to measure volume by packing. Of the approximately 50% of the sample who completed the area task successfully, about half succeeded on the volume (packing) task and half did not—there were relatively few intermediate scores. To measure volume by packing, the student must extend exactly the same packing procedure used for area into the third dimension. Clearly this is not an automatic generalisation, and it takes some time to learn. But once the idea of using the same procedure in the third dimension does occur to the student, it would seem that success follows relatively quickly.

Summary and Implications

This study has confirmed that students develop their understanding of length, area and volume measurement between Grade 1 to Grade 4 and has provided many insights into how this occurs. Although the general order length-area-volume has been confirmed, we have also obtained some findings not previously reported. In particular, it was found that students were able to measure volume by filling as well as they did length, using a similar unit iteration procedure; measurement of length was not a prerequisite for measurement of area, partly because both seemed to be affected by a general tendency towards precision in recording the unit iteration; and understanding of the unit structure for area provides the foundation for understanding measurement of volume by packing.

Some of the findings may be a consequence of the current NSW primary curriculum, where length measurement is generally taught in Grades 1-2, area in Grades 2-3, and volume in Grades 3-4. However, in the light of Reece and Kamii (2001), we were surprised to find how well students could measure volume by filling. And although most students seemed to have achieved a sound understanding of length and area measurement by Grade 4, the same cannot be said for volume (packing). It is possible that students simply do not have enough experience of packing cubes into boxes in order to make the jump from 2-dimensions to 3-dimensions in their understanding of the unit packing structure. It is also possible that teachers do not do enough to emphasise the connection between area and volume measurement.

The results presented in this paper have definite implications for the LFM (see Table 1). Firstly, a clear distinction should be made between volume (filling) and volume (packing). Since volume (filling) appears to be so much easier for students to master than volume (packing), it should be included much earlier in the curriculum. Secondly, it would appear quite appropriate to teach LFM Level 3 for length, volume (filling) and area in Grades 1-2, after Levels 1-2 have been taught in Kindergarten and Grade 1. For all three attributes, it would be helpful to emphasise the need for care and precision in recording the successive positions of the unit—a consideration which does not apply at Level 2, where multiple units can be physically juxtaposed. This is one way in which it would be advantageous to relate measurement of the different attributes to one another.

A third implication is that the measurement of volume (packing) may best be delayed until students have mastered area measurement. Level 1 for volume (packing) is identical to Level 1 for volume (filling) and could be treated as early as Kindergarten and reinforced by volume (filling) activities at Levels 2-3 in Grades 1-2. On the other hand, Levels 2-3 for volume (packing) are quite different. It would appear from our results that students need more practical experience of packing with multiple units (i.e., Level 2) in Grades 2-3 in order to gain a better understanding of the 3-dimensional array structure. Clearly a variety of teaching aids in the form of rectangular boxes and unit cubes of a variety of sizes are needed if teaching is to be effective. It may then be advantageous to delay Level 3 until Grade 4. Relating the unit structures for area and volume (packing) is a second way in which measurement of the different attributes could be beneficially linked.

References

Battista, M. T. (2003). Understanding students’ thinking about area and volume measurement. In D. Clements & G. Bright (Eds.), *Learning and teaching measurement* (pp. 122-142). Reston, VA: NCTM.

Battista, M. T., & Clements, D. H. (1996). Students’ understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education, 27*, 258-292.

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A. (1998). Students’ spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education, 29*, 503-532.

Bragg, P., & Outhred, L. (2001). Students’ knowledge of length units: Do they know more than rules about rulers? In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 377-384). Utrecht: PME.

Hubermann, A., & Miles, M. (1998). Data management and analysis methods. In N. Denzin & Y. Lincoln (Eds.), *Collecting and interpreting qualitative materials* (pp. 179-210). Thousand Oaks, CA: Sage.

Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children’s reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137-167). Mahwah, NJ: Lawrence Erlbaum.

Outhred, L., & Mitchelmore, M. C. (2000). Young children’s intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education, 31*, 144-167.

Outhred, L., Mitchelmore, M., McPhail, D., & Gould, P. (2003). Count me into measurement. In D. Clements & G. Bright (Eds.), *Learning and teaching measurement* (pp. 81-99). Reston, VA: NCTM.

Reece, C., & Kamii, C. (2001). The measurement of volume: Why do young children measure inaccurately? *School Science and Mathematics, 101*, 356-361.