Towards a Language-based Model of Students' Early Algebraic Understandings: Some Preliminary Findings

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Mathematics is a language. It has its own vocabulary, symbols, syntax and grammar. It is a powerful way of communicating ideas about the world. Increasingly, students have been encouraged to talk about their mathematics, but the potential for using linguistic features of that talk as indicators of student understanding is only beginning to be explored. This paper presents some evidence that the style of students' responses to questions about groups of similar algebraic items can indicate the level of their understanding.

Students in mathematics classrooms are encouraged to communicate their mathematical understandings in a variety of ways – by writing about their mathematics, drawing pictures or diagrams, talking with others, and explaining their ideas and thoughts. As students work together in groups or engage in discussions with the class, with the teacher or with a small group of peers, their mathematical talk has often helped teachers to make judgements about an individual's understanding. A student's use of appropriately technical mathematical vocabulary is one characteristic of student talk that can be explicitly identified by a teacher. More subtle characteristics of syntax and grammar may also influence a teacher's judgement, although these may not be made explicit.

The aim of this study is to identify characteristics of students' verbal responses which may then be used as indicators of their understandings of basic algebraic concepts and procedures. One such characteristic is the style of response students may give when asked to tell what goes on in their heads when they are asked to comment on sets of similar algebraic questions.

Background: Mathematics as a Language

> Mathematics begins and proceeds in language, it advances and stumbles because of language, and its outcomes are often assessed in language. (Durkin, 1991)

Mathematics is commonly perceived to be communicated through a written, lexically dense language. There is, however, a recent acceptance that students need to verbalise their mathematical ideas in order to describe and test relationships between mathematical identities, and hence generalise from particular instances (Dawe, 1995).

Research into the spoken language of the mathematics classroom has largely focussed on social and cultural issues influencing students' language use and ability to engage successfully in mathematics classes. It has also focussed on the ways in which teachers and students interact to create effective learning environments (Ellerton & Clements, 1996; Stephens, Waywood, Clarke & Izard,1993)

Understanding the language of mathematics does not involve simply the acquisition and use of the appropriate vocabulary, although this does seem to be the focus of some curriculum documents such as the CSF in Victoria (MacGregor, 2002) and the NSW Mathematics Syllabus 7 –10 (Board of Studies NSW, 2002). Students often struggle with the language because much mathematical language uses natural language, borrows words from natural language or ascribes particular meaning to words commonly used and has syntax not unlike, yet different in many crucial respects. These mathematical language patterns and the extent to which they affect a student's mathematical understanding have received little attention (MacGregor, 2002). There has been demonstrated a relationship between literacy levels and mathematics achievement. MacGregor and Price (1999) concluded that students who demonstrate an awareness of the structures of their natural language are more successful in dealing with algebraic statements and with translating word problems into useful mathematical forms.

How students are thinking is often more effectively revealed through their talk, if teachers realise that they need to listen to students not only to correct them, or to assess what they say, but also for how they convey their thoughts (Dawe, 1995). When recording then analysing students' talk about algebraic expressions, Smith and Phillips (2000) were cautious about the students' reasoning. Statements such as '3*x* and 3*x* is 6*x* squared' they felt, could indicate either poor understanding or sloppy expression. MacGregor and Stacey found that secondary students who used such informal, unclear, or immature ways of describing number relationships were less likely to relate those relationships to a mathematical operation (in MacGregor, 2002). Complex sentence structures, use of the logico-grammatical connectives (Dawe, 1995) such as 'so', 'if', 'because' indicate deep understanding of mathematical concepts. This understanding can be fostered by teachers providing contexts that challenge students to make mathematical arguments that in turn develop their thinking and their ways of expressing those ideas (Douek, 2002).

Listening to linguistic characteristics in student talk may also give an objective basis for judgements about students' attitude to mathematics. Bills (1999) examined the *modality* of students' responses in a one-to-one interview for clues to their response to classroom culture. Use of words such as 'may', 'might', 'just' modify the authority or confidence of a response. Rowland (2000) also examined students' use of *hedges* and teachers' use of *shields* in classroom conversations. When students use hedges, they convey a sense that they lack confidence in what they say, when teachers use shields, they provide a safety net for students to avoid giving a direct answer and so appearing to fail.

Analysis of students' descriptions of their calculations procedures has lead Bills and Gray (2001) to conclude that the style in the procedure was described may indicate the student's level of understanding.

Using data from interviews with children aged between 6 and 9 years of age Bills and Gray (2001) and Bills (2002) found that the language children use may point to individual differences in their modes of thought. The children were asked to describe 'what went on in their heads' as they performed various mental calculations. Their responses were matched with their success in correctly performing the calculations, and with the difficulty of the calculations. The results suggested that characteristics of the language used by pupils who are successful in mental calculation include the use of the pronoun 'you' rather than 'I', as well as non-particular expressions of generality in the simple present tense and as the use of logical, deductive connectives such as 'if', 'so' and 'because'.

The purpose of the study, part of which is reported in this paper, is to ascertain if such linguistic features may also be used as indicators of students' understanding of algebraic processes. In particular, this paper reports on the analysis of types of responses, whether they are focussed on particular items or are given in general terms.

Outline of the Study

Students in years eight and nine from three secondary schools in regional NSW participated. The study consisted of two parts. In the first, students were asked to complete a survey (test) of 40 algebra items, administered by their class teacher in class time. For the second part, thirty-three students who represented a range of success on the survey items, who were willing to be interviewed and who had care-giver permission were interviewed. The interviews were audio-taped and transcribed. The survey items were drawn from Stage 4 examples in the NSW 7-10 Syllabus (2002) and from Kuchemann's study (1981). They included the manipulation of algebraic expressions and the solution of simple equations. The responses were coded as correct (2), incorrect (1) and then analysed using Rasch Modelling. The interviews consisted of nine sets of questions, each set consisting of up to eight items drawn from the survey and grouped according to structural characteristics (Table 1).

Table 1
Items in Sets Presented to Students During Interviews (DD: Average Degree of Difficulty)

Set1: $3m+8+2m-5$ $5p-p+1$ $2ab+3b+ab$ $5a-2b+3a+3b$ DD: -0.473	**Set 2:** $4\times 5b$ $2ab\times a$ $4r\times 5t\times 3$ DD: -0.13	**Set 3:** $2(x+5)$ $2(x+4)+3(x-1)$ $2(x+5)+8$ DD: 1.013
Set 4a: $\frac{a}{5}+\frac{a}{10}$ $\frac{3p}{4}-\frac{p}{8}$ DD: 1.24	**Set 4b:** $\frac{4ab}{4b}$ $\frac{2}{a}\times\frac{3}{b}$ $\frac{2}{a^2}\times\frac{5a}{4}$ DD: 1.19	**Set 5:** $x+5=7$ $2t-23=49$ $5a-4=2a+8$ $x+(x+2)=(x-1)+8$ $4(p+3)=32$ $4y=20$ $10y=5$ $ax=5$ DD: 1.11
Set 6: $\frac{x}{4}=12$ $\frac{x+3}{2}=7$ $\frac{63}{x}=180$ DD: 1.49	**Set 7:** $(6xy)^2$ $(x+y)^2$ $(a-b)+b$ $8p-2(p+5)$ DD: 2.99	

The first set consisted of expressions to be simplified by adding or subtracting like terms, the third set of expressions with brackets and the fifth and sixth sets were equations to be solved. The items were written and the students were asked to act on the items mentally where possible. Students were shown an entire set of written algebraic items and asked: 'What goes on in your head when you see expressions/equations like these?' The intention here was for students to read through all items in the set and decide on some generalisation about the members of that set. The eighth set consisted of items from Kuchemann's (1981) work used to determine students' understanding of algebra syntax and symbolism. The ninth set was of questions designed to determine some background linguistic structures of the students as they answered questions that were non-algebraic or non-mathematical. These data are not included in the present analysis. Only the students' initial responses to the question have been considered.

The interview transcripts were analysed for several linguistic features of these initial responses, including whether the responses were *particular* or *general*. Particular responses are those where the students have dealt with individual items in each set without recognising any structural similarities between all the members of the set. These responses were identified in four ways: (1) One item only where the student has given a description of the process or an answer to a single item in the set; (2) one item has been responded to, but the student has offered some sort of explanation [explan]; (3) several items have been responded to, the student either giving a series of answers or a description of the algebraic process for each individual item; and (4) the student has offered some explanation for the answer or the process for each item in the set.

General responses are those where the student has seen some mathematical aspects that are shared by all members of the set. These responses are those where (1) the student quotes an all-encompassing rule; (2) the student quotes a rule and supports it by one or more examples from the set; and (3) the student gives a general procedural rule and supports it by mathematical reasons which need not rely on the particular examples presented to them [explain].

The resulting data were analysed in two ways. Firstly, in order to determine if the type of response changed as the degree of difficulty of the questions changed, the number of particular and general responses to each set was compared with the order of difficulty of each set. This degree of difficulty was established by averaging the threshold values, from the Rasch scaling, for a correct response for each of the items in the set. Secondly, the numbers of particular and general responses made by each of the students was compared to their success on the survey items in order to discover if students with differing levels of success gave predominantly different types of responses.

Results

Response Type Compared to the Degree of Difficulty of Items

Table 2 sets out the numbers of particular and general responses for each set of algebra items, according to the degree of difficulty of each set.

Set 4 was analysed in two parts, the first (4a) consisted of two expressions in which a pair of algebraic fractions was to be added or subtracted, the second (4b) required simplification of algebraic fractions to be multiplied. Students tended to deal with the first and then with the second without making any generalisations about the set as a whole.

It should be noted that the rules and generalisations did not have to be mathematically correct or appropriate for this part of the analysis. In one or two cases, students who did not score well on the survey items, and who were in the lower graded classes, quoted rules that were not mathematically useful. For example, in response to being shown set 4a, one student said:

> I think of adding those bottom numbers first, then adding the top…

Ninety-nine responses not included in the table were those considered to be *non-mathematical*. Responses such as 'I don't like fractions' or 'Run away', those which gave a general description of the members of each set such as ‘They’ve all got brackets’ or which simply repeated the question were included in this group.

Table 2
Numbers of Student Responses to Each Set of Items, Arranged by Degree of Difficulty

Set Number	Degree of difficulty	Responses to particular items in each set.					General responses to sets of items			
		1 item only	1 item + explan	2 or more in set	2 or more + explan	Total particular responses	Rule given	Rule + example	Explan	Total general responses
7	2.99	11	0	8	3	22	4	3	0	7
6	1.49	12	4	4	5	25	0	0	0	0
4a	1.24	7	1	1	1	10	3	5	1	9
4b	1.193	12	4	4	5	25	2	1	0	3
5	1.11	7	0	6	4	17	1	5	2	8
3	1.013	8	0	3	0	11	7	10	3	20
2	-0.13	5	2	7	1	17	4	6	2	12
1	-0.473	3	0	1	3	7	12	9	0	21
Totals		65	11	34	22	134	33	39	8	80

In all, 214 responses from 33 students were treated as being *mathematical*. 25.6% of all responses were considered to be general and 42.8% particular. Only 10.5% of all responses were explanatory.

Two sets, 1 and 2, were rated an average degree of difficulty less than 0, and 25.9% of all the responses were in this group. However, 40.7% of those responses were general, whilst 20.3% of the remaining responses on items with an average degree of difficulty greater than zero were general.

Response Type and Success of Students

The number of initial mathematical responses from each student varied, particularly as the weaker students often gave non-mathematical responses, even when prompted, or declined to respond to harder items such as those in sets 4a and 7. Consequently, the overall number of initial general responses by each student on each set was compared with the overall number of initial responses recorded by that student.

60% of the responses from students whose survey scores were in the top 75% of all scores were general in nature, whilst 41 % of responses from students in the bottom 25% were general. For the students whose survey scores lay in the middle 50%, only 31% of all their responses were general. The particular responses of all students were split into those responses which addressed only one item in the set and those which addressed most or all items. Students whose scores were in the lowest 25% were more likely to respond to one item only (67%) and offer only an answer or a description of the procedure:

Just go 4 times 5b, write down 20b.

Students whose scores lay between the first and third quartiles were almost equally likely to provide responses to one item, or to several (55% and 45%). However in approximately one quarter of those cases, they offered some explanations. One student’s response to being shown Set 1 was:

> …both the m’s are similar so you can add them. So go, 3m because the plus before the 2m belongs to it, so you go 3m plus 2m equals 5m and the 8 and 5, they’re both similar and the minus is in front of the 5 so it belongs to it, so it’s 3, so it would be 3m plus, no 5m plus 3. That’s how I do that question. Question 2. It’d be 5p minus p because they’re both similar terms, would be 4p, because p equals 1. 4p plus 1,that would be the answer I think. Question 5...um a and b when you put them together they are similar with the other a and b. So the plus belongs to the ab because it is in front of it. So go 2a plus ab because ab equals 1. You get 3ab so you’d go 3ab and you go plus 3b because you can’t plus b with the ab because they are different.

Of the students whose survey scores were in the top 75%, 56% responded to one item only and only 2 out of those 14 responses were explanatory, whereas 54% of the responses to many particular items were explanatory, such as in the preceding example and in the following response to Set 2.

> So, basically, multiplying 5b by 4 gives you 20b, because you are just putting 5 lots, 4 lots of 5b together. 2ab times a. Well with that you can only work with the a’s so it becomes 2 a^2b. And 4r times 5t times 3, that can become… Well you can multiply the numerals so that gives 20 tr, times that by 3, that gives you 60 tr.

Discussion

When teachers listen to students explaining their mathematical thinking, it may not be just the student’s ability to use mathematical vocabulary or to cite a rule that influences a teacher’s judgement of that student’s understanding. There are also linguistic features that are typically used by students who are more, or less, successful. One such feature is the style of response, whether it be in general terms which express understanding without necessary reference to particular items, or in more specific terms when students refer to particular items.

Two hypotheses were considered: (1) that more difficult items would attract a greater proportion of particular responses; and (2) that less successful students would be more likely to give simple responses to particular items than more successful students. In the first instance, comparison of the numbers of particular or general responses with the order of difficulty of the survey items indicated that the more difficult items were responded to in particular terms by more students and that more students responded to the easier items in general terms. This result was significant at or beyond $p = 0.001$ level. [$\chi^2 = 43.8$, with 5 degrees of freedom.]

Some of the trends may not be so clear because of the grouping of items in this analysis and the averaging of successful threshold values. For example, the third item in Set 4b was scored as either wholly correct or not correct. Many students did perform one of the simplification steps such as dividing by *a* or dividing by 2, but this did not qualify as being correct. Yet many of those students who were scored as being incorrect could still offer a description that indicated they could carry out the procedure, at least in part:

> … and question 16, I’d do the same. a^2 times 4, $4a^2$ and 2 by 5a would be 10a over $4a^2$, and … and then … the a from the top would cancel out on the bottom. The denominator would be 4a times a and then the a would cancel one of the a’s out and then it’d be 10 over 4 a …

Set 5 contained some items that were much more difficult than others because they were equations where the pronumeral of interest occurred on both sides. Many of the weaker students did not attempt these questions. Others, who were more confident in their arithmetic, successfully answered these by using trial and error methods. This meant, however, that their thinking was difficult to describe in algebraic terms, although ‘I used guess and check' is a generalisation of sorts. However, when the number of general responses to groups of items with an average degree of difficulty less than zero was compared with the number of general responses to the groups of items with an average degree of difficulty greater than zero, it was clear that the easier sets of items elicited more general responses than the harder sets.

When the responses of individual students to the sets of items were analysed, the results indicated that the more successful students were more likely to give general responses to questions such as 'Tell me what goes on in your head when you have to deal with questions like these' than students who were less successful. [$\chi^2 = 10.1$ with 3 degrees of freedom. Significant at $p=0.02$ level.] T he picture is not so clear for those students who were partly successful, that is, those whose scores were above the lowest 25% but below the top 75%. On this one criterion of response type, these students were more likely to give particular responses than those students in the bottom quartile. However, although students in the interquartile range were slightly more likely to give a response to one item only in a set, in 23% of cases that response involved some sort of explanation. Of those students who responded to most or all items in a set, 30% of the responses were explanatory. This indicated that students who are moderately successful can provide explanations for the algebraic processes they perform, although they may not yet be able to see the relationships between each item clearly, and hence may not be able to generalise.

In all but a few cases, the explanations were those which provided some procedural reasons for the algebraic process described, such as:

> You have to put them so that the denominators are the same, so that you can complete it, so the first one would be 2a over 10…

Even the best students offered little in the way of explanations which gave a conceptual justification for the algebraic procedure. One of the few such explanations was given, after a prompt, as the student responded to items in set 4a (S indicates the student's response, I, the interviewer):

> S: You have to make the denominators, the bottom one, the two, with addition you have to make the two bottom numbers the same…
> I: Why do the bottom numbers have to be the same?
> S: Um, so like the fractions are equivalent to each other.

The least successful students often gave an answer to only one item in the set, with no accompanying attempt at an explanation and proceeded no further. Those who cited a rule, even one which was appropriate, seemed unable to use it in many instances. This student, whose survey scores were in the bottom quartile, responded to items in Set 3.

> S: …I think you need to do the one inside the bracket first and then do the ones outside

This sounded like a reasonable generalisation, but I decided to explore a little further:

> I: Can you tell me more about that?
> S: um…It's like, you have to find out what x equals?
> I: [Shakes head]
> S: Oh, it wouldn't. I don't know…I wouldn't know.

One conclusion that may be drawn from such responses is that less successful students resort to quoting some sort of rule, although they may not be able to use it. More successful students, although they might focus on particular items, are inclined to act with some logic that they are able to articulate.

Each of the three schools involved in the study had graded mathematics classes, all of which participated in the survey. When matched against success on the survey items and grade as determined by the school it was found that all students whose survey scores lay in the top 75% were in the top Year 9 class in their respective schools when they completed the survey. That is, they were the most experienced students and the most successful. Six of the eight students whose scores were in the in the bottom 25% had been placed in a low Year 8 class by their schools. These represented the least experienced and the least successful students.

Linguistic features of students' explanations may serve as indicators of their level of understanding. This paper describes data that indicate that successful students are more likely to give general responses than less successful students, and that more difficult mathematical ideas evoke fewer general responses than easier items. Further examination of the data is needed to build a more robust model that could comprise several other linguistic features such as the types of pronouns used, the tense of verbs and the modality of the responses.

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