From Arithmetic to Algebra: Novice Students' Strategies for Solving Equations

Judith Falle
University of New England
<judith.falle@une.edu.au>

Students learning the art of solving equations using formal algebraic procedures are usually presented with examples that require the application of simple arithmetic knowledge. This paper suggests that such contexts do not always encourage students to use arithmetic procedures that are algebraically useful or to see the need for formal algebraic techniques. Excerpts from interviews with students reveal their thinking and some of the strategies they use. Some implications for teaching are suggested.

Translating the jostle and clamour around us into equations is half the art; solving them the other. (Kaplan, R., 1999. The Nothing That Is: a natural history of zero.)

In 1989, Fey predicted that by the year 2000 conventional school algebra would be replaced by one which used computer technology and which was grounded in realistic applications. Little has changed however in NSW secondary schools. Introductory formal algebra (Stage 4) focuses on students developing the conventions for expressions and the associated procedures for the manipulation of expressions and equation-solving skills (Board of Studies NSW, 2002, p 82 -86).

Equations express an equivalent relationship between parts of a problem. This is the first part of the 'art' of creating an expressive and useful mathematical model of some aspect of the world around us. Solving equations, however, is often seen as less an art and more a matter of following a well-worn procedural path. The reasons for the existence and meaning of equations (and expressions) remain something of a mystery to many students, at least until they have demonstrated a procedural efficiency in manipulating these algebraic creatures. The art of 'translating the jostle and clamour around' them is usually taught only after they have mastered the art of solution procedures. The conventional procedures for solution are taught and practiced with many simple, and simplistic, examples of arithmetic equations, although the syllabus explicitly recommends that

> Students need to solve equations where the solutions are not whole numbers and that require the use of algebraic methods (Board of Studies NSW, 2004, p 86)

Research into student's algebraic understandings, such as those by Demby (1997), Hall (2004) and Matz (1982), has been focussed on errors made by students. This study set out to explore the thinking of students who had correctly answered algebra items in order to determine the types of strategies they used which led to their success.

Their responses revealed that these successful students still relied mainly on arithmetic, intuitive or informal procedures which some described in loosely algebraic terms.

This paper presents excerpts from such interviews, and discusses some of the thought processes of the students and suggests some implications for teaching.

Background

Research into the thinking of students as they develop algebraic skills and understandings has examined, among other aspects, the difficulties in connecting arithmetic concepts with algebraic ideas (van Ameron, 2003), understanding and using the symbolism of algebra (Booth, 1988; Fagnant, 2002), the manipulations of expressions (Demby, 1997; Hall, 2004), the concept of the 'letters' and the solution of equations (Kuchemann, 1981) and the development of algebraic abstraction, particularly the use of *analogical reasoning* (English & Sharry, 1996).

Equations that students encounter in school are of two types – *arithmetic* and *algebraic* (English & Sharry, 1996). Many of the equations students are introduced to are of the arithmetic type described by English and Sharry (1996). These can be solved by intuitive or informal, pre-algebraic techniques such as substitution, guess and check and *backtracking*. Algebraic equations, in the sense used by English and Sharry (1996), require students to view arithmetic operators and the equality sign as relating parts of a whole rather than as instructions to act, or as indicating a closed answer, as is often the case for the equal sign (Horne, 2005). [See also Table 1.] Students also have to accept algebraic expressions as meaningful solutions to equations. They must also be able to manipulate algebraic symbols as objects in order to modify parts of equations (van Ameron, 2003). Students' use of informal techniques may be more efficacious in developing equation-solving skills than the direct teaching of formal algebraic methods (Kieran, 1989), but students have to see the need for these formal methods (Booth, 1988).

How students think about solving equations has been illuminated through analysis of their errors (Matz, 1982) and through 'conversations' (Demby, 1997). Novice algebra students make typical errors and use many informal procedures, which can work in some instances, which may be wrongly generalised to others, or which are unable to be represented algebraically. In the present study, successful students have been interviewed and their strategies for solving equations typical of those presented to students in the introductory stages discussed.

The Study

The study involved a group of twenty-five Year 9 (third year of secondary school) students in the top graded class. Along with a complete cohort of Year 8 and 9 students in two other schools, they completed an algebra survey (test) which consisted of forty items. Twelve of those items were equations to be solved [Table 1] and which have been adapted from NSW Syllabus documents (Board of Studies NSW, 2002) or from text books used by the participating schools. No calculators were to be used. The students' responses to these twelve items form the basis of the present study.

Responses to the survey items were coded as being completely correct (2), incorrect (1) or not attempted (0). The level of difficulty of each item has been determined from Rasch modelling (correct/incorrect) of responses to the items. [Table 1].

Although the students who are the focus of this paper completed the survey in Term 1 of their Year 9, they had had no further algebra lessons other than those in Year 8. They had, reportedly, met the Stage 4 Algebra outcomes (Board of Studies, NSW, 2004, p. 82 – 86). However, they seemed to be more successful at solving all or most of the equations than other top graded Year 8 students who had completed the survey at the end of their Year 8 (one term before the students in the study), and who had also completed work that would lead to their achieving Stage 4 Algebra outcomes [Figure 1]. The results of the students in the study were more nearly consistent with those of other top graded Year 9 students, at the end of Year 9 [Figure 2]. These students had completed further Algebra studies at Stage 5 level (Board of Studies NSW, 2002, p.87-88).

Success on each item by those who gave a correct response is expressed as a percentage of the number of students in each of the classes.

Table1

Survey Items used in Interviews together with Rasch Item Difficulty Levels

	Survey Number	Item	Item Difficulty Level
'Arithmetic' Equations	Items requiring simple number fact recall, integer solutions		
	27	If $x + 5 = 7$, then x = …?	-1.20
	28	If 4y =20 , then y = ….?	-0.63
	Items requiring inverse number fact recall, integer or simple fraction solutions		
	32	If 10y = 5, y = …?	0.88
	30	If $\frac{x}{4} = 12$, what is x?	0.41
	Items requiring simple, direct computation, integer solutions		
	29	What is t if $2t - 23 = 49$?	0.74
	31	Solve 4(p + 3) = 32	1.04
	33	$\frac{x+3}{2} = 7$ What is x equal to?	0.69
	35	$x + \frac{x}{3} = 4$ What is x equal to?	3.07
'Algebraic' Equations	Items requiring some algebraic reasoning and/or sound number sense		
	36	If $\frac{63}{x} = 180$ What is x?	3.37
	37	Solve: 5a – 4 = 2a + 8	2.10
	34	Solve: $x + (x + 2) = (x - 1) + 8$	2.35
	Items requiring algebraic reasoning		
	39	If ax = 5, then x = …?	3.94

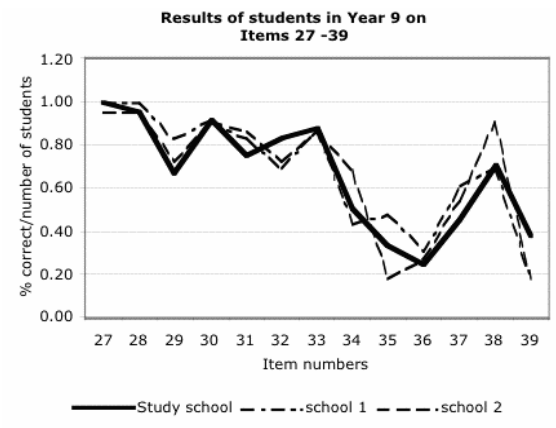

Figure 1

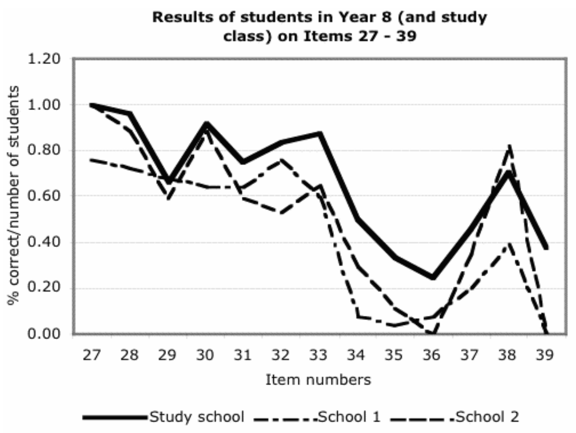

Figure 2

In audio-taped interviews after the survey, the students were asked to explain what went on in their heads as they solved the various equations.

Students were selected for interview on the basis of survey responses that gave a range of scores, their willingness to be interviewed and their having caregiver permission. All but one of the students interviewed obtained survey scores above the median when all scores were distributed across the students from all three schools. The five figure data for correct responses for the school studied were:13, 22, 28, 31, 35 (n=24) The data for the correct responses from all students in all three schools was: 2, 8, 15, 25, 39 (n=222). Detailed results for the students who were interviewed are given in Table 2.

Results and Discussion

The results of the study are presented in two forms. First, responses to the items by the students interviewed are set out in Table 2. Also included are the numbers of correctly [corr] and incorrectly [incorr] answered items and the number of non-attempts [n/a] over the whole survey. Second, the excerpts from the interviews are given to provide insight into the thinking of the students as they dealt with the different equations. Some extra equations were presented to the students during the interviews in order to clarify some points of understanding. Because of space limitations only those typical responses that serve to illustrate concisely a particular point are included. The excerpts from interviews are grouped according to the types of equations discussed. Students interviewed are identified by a number, and the interviewer as I.

Table 2
Results on Items 27 – 39 (except 38) of Students Interviewed

	Item Numbers														
Student	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**39**	**corr**	**incorr**	**n/a**
390111	2	2	2	2	1	2	2	2	1	2	2	1	30	10	0
390114	2	2	2	2	2	2	2	1	1	1	0	0	30	6	4
390104	2	2	1	2	1	2	2	2	2	2	1	1	28	12	0
390113	2	2	2	2	2	2	2	2	1	2	2	1	28	12	0
390116	2	2	2	2	2	2	2	0	2	2	2	1	23	6	11
390102	2	2	2	2	1	2	0	0	0	0	0	1	19	16	5
390128	2	1	1	2	2	2	2	0	0	0	0	1	18	12	10
390120	2	2	1	2	0	2	2	0	0	0	0	0	17	12	11
390118	2	2	2	1	2	1	0	0	0	0	0	1	13	16	11

Group 1: Equations which can be Solved Using Known Number Facts

Students were asked to describe what went on in their heads when they solved equations $x + 5 = 7$ and $4y = 20$. These equations require students to recall one fact, and the letter does not have to be operated on (Kuchemann, 1981). Horne (2005) refers to these pronumerals as 'knowns'. Very early in the interview, student 118 admitted:

Well, I just don't use the letters that much, and then, just, yeah. I don't work with the letters that much.

Other students described the process as:

> 113:Because of the plus symbol, *x* plus 5, so you go up numbers until you reach 7, that's 2.

Other students described their thinking:

> I: Why did you say 7 take away 5? [Q27]
> 118: because that's the number you need to find 5 plus x equals 7, so what's in between kind of … Because if there are two numbers and you take one of them from the answer, then there's going to be the other number, what you need.

and

> 116: Well, twenty-eight's [$4y = 20$] pretty easy because I know 4 times 5 equals 20 so *y* equals 5.

Typically, when solving these equations, students rely on known number facts to arrive at an answer. This type of response cannot be written as an algebraically useful representation. Nor does it provide an analogy that can be used to deal with questions like those in the following group.

Group 2: Equations which rely on Understanding of Arithmetic Relationships

I: What if I gave you a question like that: 6*y* equals 7? Tell me what you would do?
116: um …
I: So, what are you thinking?
116: I'm trying to find what they are. 6 from 6, no, … I don't know. I was trying to find what a sixth would equal, but that wouldn't work anyway.
I: A sixth? A sixth of what?
116: Like, I dunno, It wouldn't work anyway, so.

This student recognised that the answer would have to be one sixth of something, but two points are important here. Although he explained his reasoning in solving $4y = 20$ as 'you divide 20 by 4', it would appear that the strategy actually used was to identify the known product of 4 and 5, but then rephrase the reasoning in classroom terms, as the following student made explicit.

102: [reads] If 4*y* equals 20 then *y* must equal… [Q28]. Well here, I know that 4*y* is the same as 4 multiplied by *y*, so 4 multiplied by *y* equals 20. Then to find *y* I divide 20 by 4.

However, in the equation $6y = 7$, because there was no known product of 6 and another integer to give 7, the student was unable to support the result of using the arithmetic inverse, an algebraically useful concept. Secondly, as the answer was a fraction, not an integer, it was clearly one with which the student was uncomfortable. It is worth noting here that the syllabus makes explicit that students need to have dealt with such equations by the end of Stage 4 (Board of Studies NSW, 2004, p. 86).

Some students used quasi-algebraic strategies that explicitly relied on seeing arithmetic relationships to deal with equations such as $t – 48.4 = 201.9$.

I: So, what goes on in this one, *t* take away 48.4 equals 201.9? What would you do?
104: 48 point, aahh, I'd do 201.9 plus 48.4, equals *t*.
I: Can you explain to me why?
104: I'm reversing it to find out. If I'm attempting to find out *t*, I'm not. I can just reverse it. So, I can, instead of finding out two hundred and one point nine I can find out *t*.

However when asked what reversing does to an equation, this student replied:

104: Nothing. Well, it reverses…

The idea of 'reversing it' contains seeds of the understanding of 'inverses', but it is not clear what it is that is being 'reversed'. As one other student explained when presented with questions 30, 33 and 36 [Table 1]:

111: Well you have to change it around again, like do the opposite. Like it would be. 4 times, 4 divided by. 12 divided by 4, is *x*. [Q30]. You have to times 12 by 4 to get what is over 4 …

It is not clear what 'the opposite' is, although this student did get the correct answers to each of these questions [Table 2].

The confusion of 'opposites' or 'reversing' equations or operations with no idea of the mathematical relationship between the parts of the equations is vividly illustrated as the student uses the surface structure of the equation to reason analogically, focussing on the one feature – that of one number being divided by another:

111: You have times 7 by 2 and minus 3 from that to find out what x was, in 33. And 36, it would be 180 divided by 63.

Equations such as those in questions 31 and 33 [Table 1] can be solved by students 'closing' each arithmetic step.

120: um … aahh, so you'd go 7…divided by. No 7 times 2, equals 14. So 14 minus 3 is 11. So it's 11. *x* equals 11?

Sometimes, the 'closure' is not so complete, at least in the explanation.

I: How would you go about doing question 29?
113: I don't know. I'd be adding 23 plus 49 and getting that answer and dividing by 2.

The more successful students could deal in a similar way, by closing on each step, with an equation such as the following. (The use of the '÷' sign suggests other issues worth further investigation.)

I: Tell me the steps that you would go through to rewrite *x* plus *a* over *b* equals *c* [(x+a)/b = c].
104: Well, first of all I would change it from a fraction to a normal equation, division, and I'd just bear in mind that *x* plus *a* over *b* is the same as writing *x* plus *a* divided by *b* [÷], so then I'd write …
I: See if you can do it without writing.
104: OK. *x* plus *a* divided by *b* equals *c*, therefore *c* times *b* equals *x* plus *a*, therefore, *c* times *b* equals *x* plus *a*, … therefore *x* equals *c* times *b* minus *a*.

Group 3: Equations Requiring Explicit Algebraic Strategies

When students were presented with the algebraic equations in questions 37, 34 and39, they used some algebraic strategies to transform the equations and then used a 'guess and check' process.

104: Well in this one [Q34] I simplify both sides first. 2*x* plus 2 equals *x* minus 7… so if 2*x* plus 2 equals *x* minus 7 that means that … [writes]. Doesn't make any sense.
I: Why not?
104: Because 2*x* plus 2 can't equal *x* minus 7
I: Why not?
104: Because we are saying two lots of a number plus 2 can't equal that same number minus 7 […]Oh, wait a minute, it's plus 7, it can work …Yeah, I did it by guess and check I think. I don't know how to work them out, properly.

Another student was presented in the interview with: $16.5 – 7.3x = 14.2x – 4$

I: What about the first one on that sheet. 16.5 take away 7.3*x* is 14.2*x* take away 4?
116: To find *x*? um ….
I: What are you thinking?
116: Trying to think of a way other than just trying different numbers for *x*

The students have realised the need for strategies that are more efficient than the informal 'guess and check'. To use iterative strategies effectively, the students need to have a sound number sense. For example:

120: …63 over *x* equals 180 [reading]. … It's just 180 times *x*. First to find it out you've got to go 63 divided by *x* equals 180, so 180 times *x*, I mean, 180 times *x* equals 63, I think.
I: Therefore, what's *x*?
120: oh! … 180 times, oh, aahhh … I'm not sure, so I'm doing 180 times *x* equals 63. So it's, it's around like a third, around a third. So 63 divided by a third. Well it has to be lower than zero, because then it will go into that more than what that number is to get 180. So it has to be around a third I think.

When such a number sense is lacking, the hunt for the 'right number' takes considerable time (and a calculator).

102:…and then the next one is 63 …[Q.36]. Firstly I've got to find the bottom, the number below the fraction, which would, can't figure it out without a calculator off the top of my head.
I: What would you ask your calculator to do there?
102: I'd probably go just 63 and I'd start, I'd probably start, usually I start with um, any number, say I start with maybe a three, and then if its absolutely, a number which is very low then maybe I'd try 63 over 10, if it was a number which was too high then I'd work my way back between something that has to be between three and ten. So it takes a while, but I usually I find that the most reliable way for me for doing it.

However, when presented with question 39 [*ax* =5] the arithmetic strategies, including guess and check failed. All of the students interviewed resorted to guess and check when dealing with equations that would have required them to use algebraic manipulations if they were less simple, and none successfully answered question 39. Most answered that *x* was 2.5, 1, 5 or 4. All of them, on the other hand (except student 118) answered $10y = 5$ successfully. One student however found it impossible to separate the process and the product:

> 128: If 10*y* equals 5, *y* … … um *y* equals.. divided by 2.
> I: What is divided by 2?
> 128: 10, 10 divided by equals 5, so *y* equals divided by 2.

Conclusion

The responses above provide some insights into strategies adopted by successful students to solve typical introductory equations and the mathematical 'usefulness' of those strategies and raise many points about students' mathematical understandings as they are introduced to formal, school algebra. These were, by several measures, competent mathematical students. They were in a top-graded class, scored well on an algebra test of uncomplicated, but typical items, and demonstrated a generally sound number sense and a facility with number facts.

However, their explanations gave little evidence that they were using formal algebraic techniques to solve equations, although they had been well-rehearsed in those elementary techniques. Their Year 8 (Stage 4) teacher had demonstrated the procedures and given the students ample practice through textbook exercises. Because of the nature of the equations in the survey they could use their knowledge of number facts, arithmetic relationships and recursive methods such as 'guess and check'. These methods appeared to work for some students even for questions 37 and 34, possibly because the numbers involved were small positive integers. During the interview, even those students who had correctly solved those equations in the survey could not do so. It was at this point that these students recognised the need for other, more efficient methods to solve some equations. This is the point where teaching of formal techniques can begin.

Some students could transform simple one- or two-step literal equations, using arithmetic analogies. However, when they had to deal with making *x* the subject of equations of the form $a/x = b$, it was clear that in previous examples they acted on the surface similarities and 'did the opposite', rather than perceive any meaningful mathematical relationships, and hence correctly use inverse operations to transform the equations. Transformation of equations that are not transparently arithmetic demands that students understand the relationships between and within terms. This relational thinking is at the centre of algebraic understanding. The conventions of formal school algebra derive from a generalisation of arithmetic relationships and arithmetic procedures. But those relationships have to be made explicit and the procedures have to be generalisable (Booth, 1988). Generalisation can only develop from a broad range of experiences. So students need to encounter, early on, equations which have other than small positive integer coefficients, or solutions. Equations which use the entire set of rational numbers help students develop basic arithmetic skills, good number sense and encourage the use of strategies that are more efficient and generalisable. This is consistent with Stage 4 syllabus expectations (Board of Studies NSW, 2004).

Presenting students with simple equations, which they can solve by arithmetic means, yet insisting they use formal algebraic techniques does not encourage the development of algebraic understanding. Such examples dominate many textbook exercises. In these simple cases, equation solving need not depend on structural perception of equations nor on the correct manipulation of the equation (van Ameron 2003). Using number facts, using arithmetic inverses and solving equations step by closed step, selecting numbers and 'homing in' on an answer are useful, appropriate strategies. Yet, by exposing students to arithmetically difficult examples the cognitive demands on the student are increased. Not only do they have to cope with the new, formal concepts of algebra, they have also to cope with opaque arithmetic.

Perhaps the students in the study were 'doing algebra', but not 'thinking algebraically'. They had demonstrated that they could solve simple equations by providing, in most cases, correct answers. By exploring their thinking, it became apparent that the correct answer may often mask incorrect, or algebraically inappropriate thinking. Thinking algebraically considers relationships between mathematical objects and the consequences of acting to change those relationships. If we are to teach formal algebra, we need to develop the students' facility with the processes, their conceptual understanding of the structures and relationships between numbers and their strategic sense of the best 'algebra' to use in particular context.

But, do we need to teach formal algebra at all? If so, to whom? Fey's technological vision (1989) has not yet come to pass, but the software that transposes and solves complex equations is available and increasingly accessible. The skills students need in order to use that software may not turn out to be the skills embodied in formal, conventional algebra as is presently taught in school. However, the skill to translate the 'jostle and clamour around us' into effective and useful algebraic models will still be needed. This implies that the present focus on developing the procedural skills of algebraic manipulations may need to shift to one on developing students' abilities to understand, interpret and represent problems in algebraic ways.

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