The Growth of Schematic Thinking about Derivative

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Since our conceptual structures are a major factor in learning, it may be hypothesised that the richer these schemas are the better the learning that will result. This paper reports on a study of students' understanding of derivative, and the thinking they construct. It follows the progress of two students, James and Bob, and describes a snapshot of the richness of their thinking in this area. This is related to a framework of knowing proposed by the authors, and examples of the possible value of schematic extensibility in terms of understanding new ideas

The notion of schemas has gained considerable support in the literature as a metaphor for the manner in which cognitive structures are formed and mature (Skemp, 1979; Anderson, 1995). Such schemas are formed as students' experiences in a conceptual area expand, and actions, processes, and objects are linked into coherent structures (Dubinsky & McDonald, 2001). It seems to be reasonably evident that an individual's existing conceptual structures, or schemas, are a key determinant of ability to understand and hence make progress in learning, either by promoting or restricting the association of new concepts. The richer the schemas, in terms of the spread of the network, the qualitative nature and the strength of the links between the constituent parts, the more likely they are to support such expansion. However, capturing how schemas change in any knowledge domain is very difficult. The study reported here attempts to describe the qualitative nature of the growth of two students' schematic structures surrounding the concept of derivative.

One key factor built into schemas for a given concept is the representational basis of the concept. Representation cannot be divorced from the process of mathematical understanding since the ability to represent is implicit in learning mathematics. It is not possible to think consciously about mathematics without using some form of representation. For example, symbols (including words) are used to represent mathematical objects, processes or structures (i.e. mathematical concepts), diagrams are constructed to make sense of relationships involved in information given in mathematical problems, and graphs are drawn to provide visual support to properties and behaviour of functions.

Research has shown that there is a constitutive relationship between students' representational abilities and their mathematical understanding and problem solving proficiency (Cifarelli, 1998; Lesh, 2000). For example, learners' emerging understanding can be attributed to their capability to represent a problem in a number of different ways, allowing them to approach solutions from different perspectives (Sigel, 1999). Studies in algebra and calculus (e.g., Orton, 1983; Heid, 1988) have shown significant improvement in students' performance when taught in multiple representational environments. Based on a study examining understanding of differentiation, Heid's (1988) investigation also showed that the performance of calculus students who were exposed to meanings and concepts first, using a variety of representations, followed by an emphasis on skills, performed better compared with students taught in the reverse sequence. Slavit (1996, p. 14) noticed how for 16 year old algebra students the "multi-representational capabilities of the [graphic calculator] allowed additional aspects of a problem to be quickly analysed in a 'representationally-connected' fashion." Such studies suggest that conceptual development of students could be enhanced through teaching and learning with multiple representational perspectives. A suggested reason for this is that a multi-representational approach may divert the focus of attention from the representation, through the abstraction and identification of links between representations, to the concept that it represents (Noble, Nemirovsky, Wright, & Tierney, 2001). Thus, the formation of integrated multiple representations for the same phenomenon might encourage meaningful understanding, enhancing representational fluency (Lesh, 2000). It has to be recognised though that student use of multiple representations may not be automatic. For example, Crowley (2000) noted that graphical and symbolic representations were not linked by students unless they were explicitly asked to do so, and Weigand and Weller (2001) found that students often lacked the patience to read, interpret and reflect on different representations. Further, Kendal and Stacey (2002) found that only the most capable students achieved the goal of developing facility with numerical, graphical and symbolic representations of functions and derivatives.

In this paper, we characterise a snapshot of two students' mathematical knowing vis-à-vis their representational abilities. This characterisation is based on the previously described Representational Framework of Knowing Derivative (delos Santos & Thomas, 2003, p.326). Presented in matrix form, this framework maps students' dimensions of knowing across their representational preferences, with each cell describing possible representational abilities, as they engage in solving problems. The dimensions of knowing are categorised into procedure-oriented, process-oriented, object-oriented, concept-oriented, and versatile, characterised according to different modes of representations (symbolic, graphical, and numeric). Due to space limitations we present here only descriptions for the last two dimensions of knowing.

- *Concept-oriented knowing* – the level where the learner has created a 'bigger picture,' comprising schemas containing procedures, processes, and objects arranged in a relational manner. The learner with concept-oriented knowing can provide answers to why certain procedures and processes work, is able to create conceptual links across representations and relate process and object tools used in problem solving.
- *Versatile knowing* – the learner has sufficiently wide range of the four types of knowing to enable choice in problem solving, along with sufficiently developed metacognitive ability to choose an appropriate perspective at any given point in time, and the ability to move fluently between the chosen perspectives as required.

Method

This research comprised case studies of James and Bob (pseudonyms), two male Form 7 students (aged 18 years) from a high-level socio-economic private school in Auckland, New Zealand. The analysis of the two students' thinking and understanding forms part of a study of the understanding of derivative which took place in four schools where the teachers agreed to 'integrate' graphic calculators in their teaching of calculus. From the student volunteers three or four students from each school were selected as representative of high-, mid-, and low-achievers, using a pre-test. The two high-achieving students described here were interviewed before and after the intervention, which comprised a module of work on derivative using TI-83 calculators. After the module, a post-test was given followed by the post-intervention interview. The pre-intervention interviews were video-taped, while the post interviews were audio-taped, and both were transcribed and the data analysed, together with the test results. In addition, there is analysis of concept maps prepared by the students (considered to be an externalisation of conceptual schemas) and interview data, including interpretation of familiar and unfamiliar symbolisations related to derivative. The two students were taught by a teacher very experienced at ‘integrating’ graphic calculators into her teaching. She has a strong belief that an emphasis on multiple representation, supported by the use of the graphic calculator in teaching and learning provides better opportunities for conceptual learning.

Results and Discussion

On the pre-test James obtained 51% and Bob 58% (class average: 34.8%), while in the post-test, Bob got 77% and James scored 76% (class average: 51.2%), placing them among the top three participating students in their calculus class. In view of this we expected them to exhibit evidence of relatively rich schemas for derivative. What we found from the interviews is described below.

The Concept of Derivative

There was some evidence from the students that their conceptual structure for derivative was changing, even during the eight weeks of the study. This was inferred from the differences in the first and second interviews and the concept maps. In the first interview Bob was unclear on the connection of the derivative to gradient:

> B. I don't actually know, we haven't been taught why it is that, but it comes back to if you've got y, or you've got a function, perhaps a function of f, in terms of x, and perhaps it's used x^2 for reasons I don't actually know, but the gradient comes from, and comes from being $2x$ and that comes from…

And immediately after this comment he resorted to a procedural explanation of how to find the derivative:

> B. So what you are basically doing is you take the power, and then you multiply, you take turn from [sic] the coefficient the number that the x gets multiplied, then you subtract 1 from the index of the power. …It’s the equation or function you work out when you want to find the gradient, the original function. So the trick that I remember was…if you got the function of x is ax^b, then the derivative of that is bax^{b-1}…that’s sort of what I remembered about derivatives.

At this point, during the first interview, James also described derivative in a manner betraying a procedural tendency, and a lack of rich conceptual understanding:

> J. Derivative of a function, you get…after you differentiated?...and you differentiate by first principles and you can proceed by rule…and magically we know that that way…when you derive something, you will get the gradient but I can’t remember the…there is a pretty good reason…there is a reason behind everything.

In their responses both seemed to view derivative as a function obtained by differentiation, and subsequently both showed reasonable differentiation skills. However, they did not elaborate regarding different representational modes. They may have had some mastery of differentiation procedures and an object-oriented view of derivative, but their conceptual understanding had not yet matured. For example, though James knew some elementary rules of differentiation, and knew that derivatives can be obtained by first principles, he was limited to a description of surface features of the first principles expression, and could not correctly recall it, saying that “it’s all over x” and that the limit was “x approaching zero.”

In contrast, when James and Bob were asked, in the second interview, to explain their understanding of derivative, they gave the following responses:

J: Well, derivative is sort of like …it is the graph of the gradient of this other graph…I think it's more a fact that it's the gradient at a given point on the graph. That makes more sense when I think about it. So, yeah, but the graph gets you just all of these with the corresponding *x*-*y*...makes it a lot easier to read. …derivative of a function you get…after you differentiated. You differentiate by first principles, and you proceed by rules.

B: The first derivative…is that how much of the gradient of that tangent to that point is. So, if you draw a straight line through that point with the gradient equal to it [the derivative], that will be the gradient of that line…By doing the *proper* process…you take the limit of the gradient…[starts describing the formulation of first principles]…that's how we worked out the gradient at that very particular point. Then we were taught to *cheat*, which was to play with powers [and demonstrated the rule for the derivative of x^n—italics added]

Both students relate derivative to graphs, particularly to the gradient of a tangent 'at a given point on the graph' and 'to that point'. Such a pointwise approach has been characterised by Thomas (2005) as a process perspective. In turn, their description of derivative has both a graphical basis and an appropriate algebraic component. They both presented a derivation of the gradient (derivative) from first principles, and offered another method to obtain the derivative rules (*cheat*, as Bob describes it). While low-achieving students may describe derivative in a primarily procedural manner, using only a symbolic representation (delos Santos & Thomas, 2003), these two students have demonstrated a more complex description, employing several representational perspectives and they appeared to be using these as cognitive tools. They constituted the definition of derivative within both the graphic and the symbolic modes, and described derivative as an object (the gradient of a point) that could be seen and thought of graphically, and as a process, through its first principles derivation using symbols and its application to particular points. These were underpinned by relevant links between the process and its graphical representation. Moreover, in both interviews, they appropriated another crucial interpretation of derivative, namely as a rate of change and/or a function in itself.

J1: They're all functions in their own respect…It's…rate of change. J2: …this is the rate of change with respect to *y*, which is the same as the gradient.

B1: …involves how much things change over a period of time…there's a rate of change. B2: …this one is the rate of change, and it's also the derivative.

What is evident in their responses is not just their facility in describing derivative using symbols and graphs, but also the links they have built to attribute meaning for derivative. It seems that the graphical representation has served as a cognitive tool in building a link between representations, and hence a multi-representational perspective of derivative.

Changes in Concept Maps

The changes in James' and Bob's conceptual understanding of derivative and differentiation between the two interviews described above is supported by their respective concept maps of derivative and differentiation (see Figures 1 and 2). In his first concept map, James' presentation of ideas was linear, tabular, compartmentalised and hierarchical, with function on top, differentiation executed downward and anti-differentiation returning to the function.

The second concept map was transformed into a circular form, with function in the centre and other ideas emanating from it. He was, however, quick to note that the derivatives are also functions, seemingly emphasising that the centre refers to a specific function, and that the circular map is embedded in family of functions. Bob's first concept map was more of a collection of propositions, which was transformed into a more complex map with web-like links centring on the idea of derivative. Their second concept maps still contain symbols, and graphically-based ideas such as gradient, turning point and concavity, but are richer in terms of links to concepts such as rate of change, and integration. These qualitative differences in their concept maps appear to indicate both growth and transformation of their continuous conception of derivative. A limitation of these maps is the failure to have the links labelled. Thus it is not possible to differentiate qualitatively between, say, a link from derivative to rate of change that was thought of as "is a" (object link) and one that was "is used to find" (procedural link).

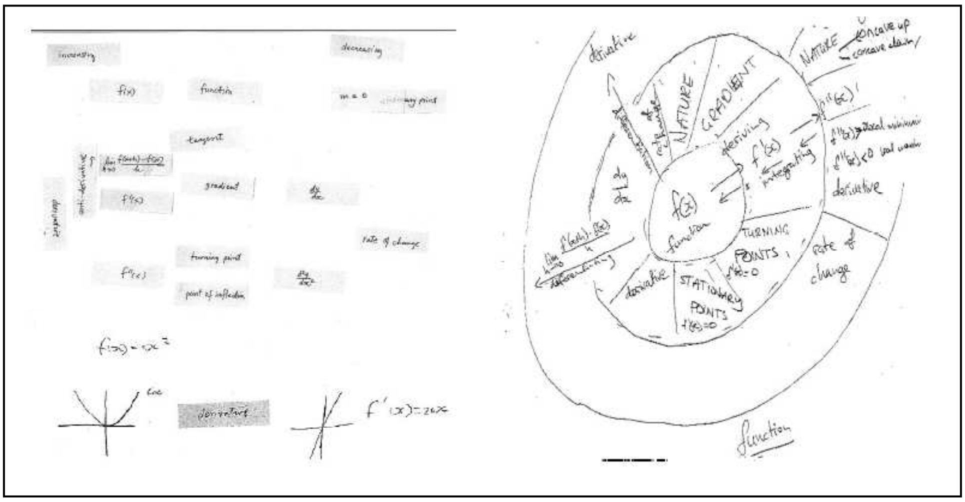

Figure 1. Bob's first and second interview derivative concept maps.

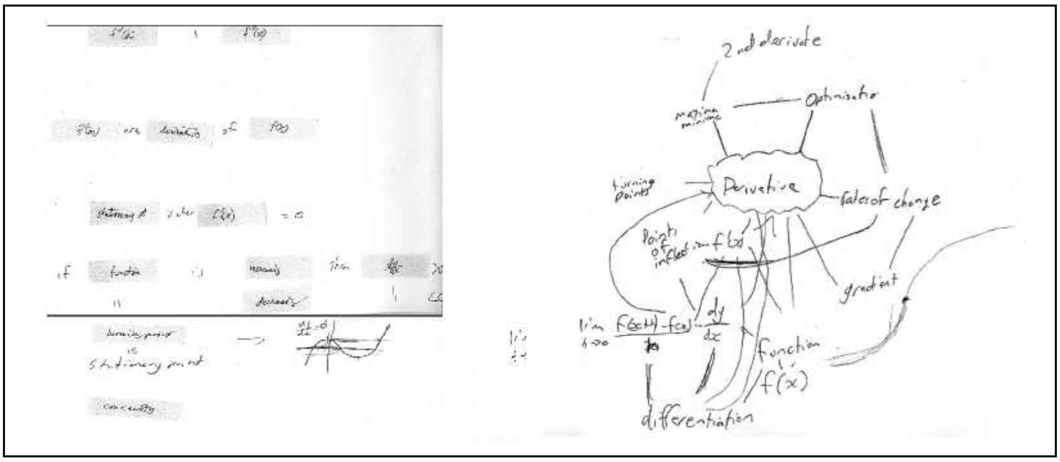

Figure 2. James' first and second interview derivative concept maps.

Using the Rich Derivative Schemas

One of the expected benefits of a rich schema in any part of mathematics is that it is more readily extensible. When new ideas, possibly represented by unfamiliar symbols or contexts, are encountered they are more easily assimilated. One of the methods we used in this research was to present the students with familiar symbols for derivative in either less familiar or unfamiliar contexts to see their reaction. Examples of these symbolisations were $\frac{d\left(\frac{dy}{dx}\right)}{dx}$, $f(f'(x))$, and $f'(f'(x))$. For the first symbol, during the second interview Bob said "that is the second derivative and that is how much the gradient of the function is changing " and James called it "the second derivative of the function *y*". They were both immediately able to allow the process to operate on the derived function, and to link it to $\frac{d^2y}{dx^2}$. When James initially tried to interpret the meaning of $f(f'(x))$ he displayed the ability to think of the symbol $f'(x)$ as an object, which he described as the derivative function. He read the symbol as "the first function of the derivative function." In his attempt to describe what the symbol meant, he used a specific family of functions of the form $f(x) = x^n$. His work, where he used $f(x) = x^2$, obtained $f(f'(x)) = (2x)^2$, and worked from there, is shown in Figure 3.

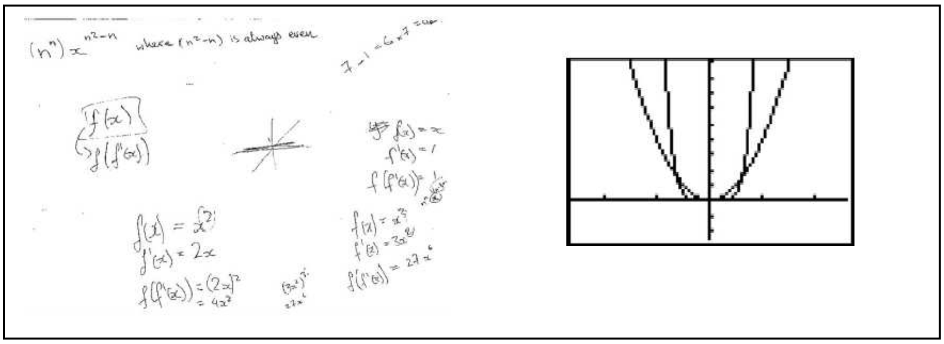

Figure 3. James' working on $f(f'(x))$, by-hand and on the GC.

He then picked up the graphic calculator to graph the resulting functions saying that "It's going away…it's always gonna be steeper than this original function…it's gonna be steeper…it's also gonna be concave upward." After some time, while comparing the algebraic results with the graphs, he generalised the result to $x^n x^{n^2-n}$, recognising n^2–n as always even. Bob, however, immediately formalised the notation, describing $f(f'(x))$ as a composite function, and worked on it as such: "that's the tangent value of the similar function on the gradient…it is the result you get from…the function *f* of the gradient of the derivative of that function." Though the response sounds vague, he demonstrated how to obtain the result using a specific function, $f(x) = x^2$, "if we take $f'(x) = 2x$, you'll end up…to $4x^2$." When asked what the significance of the symbol is, he replied "to generate a new function. That's what I see from it." Though both James and Bob saw the symbols as a composite function, Bob is more locked into a process view, whereas James went further and was able to generalise the outcome from the process, symbolising it and describing the result as the generation of a new function. Again, what appears to be interesting here is the way their attempts to make sense of the symbols employed several interconnected representations. They were able to relate the symbols to words describing concepts, and to others that were graphical in nature, namely tangent and gradient. James in particular was able to use the graphic calculator to reason from a graphical perspective.

When asked to describe $f'(f'(x))$ James said “that does imply second derivative”. Hence instead of applying the composite function thinking he had used only seconds previously he saw this as the second derivative $f''(x)$. This could be the result of a strictly linguistic interpretation of the symbolism. Reading $f'(x)$ as *f*-dashed of *x*, may cause one to read $f'(f'(x))$ as *f*-dashed of *f*-dashed of *x*. This in turn leads to James statement that “It’s the derived function of the first derived function.”, and hence the second derivative. When asked about $f'(f'(x))$ Bob responded “That is yet another one... you'll be working out the gradient of the value, which is equal to the gradient when you've got a number which is *x*...in a circle, which I don't actually think it means…you can definitely evaluate it the number I don't think it means much.” Clearly he was confused, and not able to apply his understanding of composite function to this symbolism.

Approaching Versatility

Both these students have moved from a mainly procedural perspective to a more concept-oriented view of derivative. They have increased the number of links in their derivative schema, particularly with respect to increased representations. However, while they have relatively rich schemas of derivative that can be described as *concept-oriented-knowing* in our framework, they are not yet *versatile* in their conceptions. One reason may be that a possible contributory factor in the ability of both students to develop a conceptual understanding of derivative, namely their *preference for thinking graphically*, may be holding them back. A feature of versatile knowing is the ability to control perspectives and to move seamlessly between representations as and when required in mathematical thinking and problem solving. When asked for the use of derivative, for example, Bob pointed out the problem of “looking for the point of a minimum of something, by finding the minimum of a *curve that models the situation*, and evaluating the minimum needed.” With reference to the curve, he described that it could be done by “solving for the derivative equal to zero,” and started to describe how “that’s the turning point.” He further explained how “that [the turning point] tells you whether it’s a maximum or a minimum…by checking the derivative on either side to see if the sign [of the derivative] changes from negative to positive.” Such comments show a procedural orientation but also demonstrate how the decision-making part of the problem is linked to a graphical interpretation of results.

James, in describing turning points, explained in terms of mathematical ideas, not just procedures, that “you have to investigate because sign is zero. So, investigate the sign of $f'(x)$…if the sign is positive here, because the sign is increasing and [referring to the other side of the turning point] decreasing, getting more and more…cause it is zero here, it gives us a turning point…it implies a positive on this side.” However, his view is still graphically oriented. This was also true when he was queried about real-life applications of derivative, he replied “… sort of gotta think of graphical.”, and at another point in the interview, he commented that “I think algebraically, using graphs and stuff…and answers algebraically using the graph really comes in very handy.”

It is not possible to ascribe the positive changes in thinking of Bob and James in this snapshot to any particular item or activity. Clearly the growth of schemas is organic and affected by many variables. However, both James and Bob had used the GC in a number of ways. Bob said he used it to check answers, and he wrote little programs, including one for a parabola. He felt that it helped him to “get a better picture of it sometimes through the graph…See what's actually happened, why it happened and how it happened.” When asked if the use of the GC by the teacher helped him understand better Bob replied “In terms of understanding I think it has helped…it allows us to see what the answer is.” However, he had some reservations on the GC use, noting that “it would possibly make me lazy with working pressing buttons, getting the calculator to do it so forget the working out”. It may be that the use of the GC was a factor in their progress, and if so this is likely due to teacher privileging (Kendal & Stacey, 1999) of GC use by their experienced teacher, and the multi-representational approach she adopted with the GC. However, results on this are inconclusive and further exploration of any link is required.

References

Anderson, J. R. (1995). *Cognitive psychology and its implications* (Fourth edition). New York: W. H. Freeman & Company.

Cifarelli, V. V. (1998). The development of mental representations as a problem solving activity. *Journal of Mathematical Behaviour, 17*(2), 239-264.

Crowley, L. (2000). *Cognitive structures in college algebra.* Unpublished doctoral dissertation, University of Warwick, UK.

delos Santos, A. G., & Thomas, M. O. J. (2003). Representational ability and understanding of derivative. In N. A. Pateman, B. J. Dougherty & J. Zilliox (Eds.), *Proceedings of the 27th Conference of PME* (Vol. 2, pp. 325-332). Honolulu: University of Hawaii.

Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning, In D. Holton (Ed.) *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 275-282). Dordrecht: Kluwer Academic Publishers.

Heid, M. K. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education, 19*, 3-25.

Kendal, M., & Stacey, K. (1999). Varieties of teacher privileging for teaching calculus with computer algebra systems, *The International Journal of Computer Algebra in Mathematics Education, 6*(4), 233-247.

Kendal, M., & Stacey, K. (2002). Teachers in transition: Moving towards CAS-supported classrooms. *Zentralblatt für Didaktik der Mathematik, 34*(5), 196-203.

Lesh, R. (2000). What mathematical abilities are most needed for success beyond school in a technology based age of information? In M. O. J. Thomas (Ed.), *Proceedings of TIME 2000 an International Conference on the Technology in Mathematics Education* (pp. 73-83). Auckland, NZ: The University of Auckland & AUT.

Noble, T., Nemirovsky, R., Wright, T., & Tierney, C. (2001). Experiencing change: The mathematics of change in multiple environments. *Journal for Research in Mathematics Education, 32*(1), 85-108.

Orton, A. (1983). Students’ understanding of differentiation. *Educational Studies in Mathematics, 14*, 235-250.

Sigel, I. E. (1999). *Development of mental representation: Theories and applications.* New Jersey: Lawrence Erlbaum Associates.

Skemp, R. R. (1979). *Intelligence, learning and action—A foundation for theory and practice in education.* Chichester, UK: Wiley.

Slavit, D. (1996). Graphing calculators in a “hybrid” algebra II classroom. *For the Learning of Mathematics, 16*, 9-14.

Thomas, M. O. J. (in print) Conceptual representations and versatile mathematical thinking (Proceedings of ICMI-10), Copenhagen, Denmark: ICMI.

Weigand, H-G., & Weller, H. (2001). Changes of working styles in a computer algebra environment—The case of functions. *International Journal of Computers for Mathematical Learning, 6*, 87-111.