Year 6 Students' Methods of Comparing the Size of Fractions

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The aim of this study was to gain a better understanding of the ways in which children conceptualise fractions as quantities when they are asked to compare the size of fractions. One hundred children aged 11 and 12 were asked to choose the larger of two fractions and to explain their reasoning. The questions progressed from comparing unit fractions to comparing fractions close to one-whole with different denominators. By coding the children's responses according to the comparative strategy evident in their explanations, a range of strategy categories was identified and exemplified.

When students encounter problems involving fractions, they often solve the problems by using their knowledge of whole numbers (Behr, Wachsmuth, Post & Lesh, 1984; Mack, 1995; Yoshida & Kuriyama, 1995). With common fractions students may reason that $\frac{1}{8}$ is larger than $\frac{1}{7}$ because 8 is larger than 7, or they may believe that $\frac{3}{4}$ is the same as $\frac{4}{5}$ because in both fractions the difference between numerator and denominator is 1 (Behr, Wachsmuth, Post et al., 1984). This reasoning appears to be based on students continuing to use properties they learned from operating with whole numbers. "For all children, their previous whole-number schemas have influenced their ability to reason about the order relation for fractions" (Post & Cramer, 1987).

When comparing fractions, students often focus on the whole number numerators, the whole number denominators or both whole numbers. In the Concepts in Secondary Mathematics and Science (CSMS) project, approximately one-quarter of 14 and 15 year-olds responded with $\frac{1}{3}$ when asked to write down a fraction that came between $\frac{1}{2}$ and $\frac{2}{3}$ (Hart, Brown, Kuchemann, Kerslake, Ruddock & McCartney, 1981). Such a response is consistent with comparing both numerators ($\frac{2}{3} > \frac{1}{3}$) and denominators (3>2) as whole numbers.

Vinner (1997) described similar fraction comparison strategies, such as "the bigger the denominator the smaller the fraction", as pseudo-analytical behaviour. The application of these spurious or incomplete strategies can result in correct answers for the wrong reasons. A question from the Third International Mathematics and Science Study (TIMSS, 1996), "Which of the following numbers is the smallest? (a) 1/6 (b) 2/3 (c) 1/3 (d) 1/2" could be correctly answered using "the bigger the denominator the smaller the fraction". However, the application of a similar strategy, "the smaller the denominator the greater the fraction", to another TIMSS question, "Which number is the greatest? (a) 4/5 (b) 3/4 (c) 5/8 (d) 7/10" would lead to the selection of option (b). Vinner reported that 42.1% fewer 8th grade Israeli students correctly answered the second question than the first question. Approximately 39% of the students selected option (b), which may have been influenced by the application of a strategy such as "the smaller the denominator the greater the fraction". The difference in the percentage of correct responses to these two similar fraction comparison tasks suggests that the expected method of using equivalent fractions may not have been the only substantial coherent method applied.

Thinking quantitatively about fractions relies upon equal-partitioning (Lamon, 1996) and the invariance of the whole (Yoshida & Sawano, 2002). In representing a number less than one, magnitudes of one as a whole should be of the same size to compare fractions. The invariance of the whole is essential in comparing what Yoshida (Yoshida, 2004) describes as quantity fractions. Quantity fractions are those that reference a universal unit-whole that is independent of any situation and allows one to answer questions such as, which is larger, one-half or one-quarter? In dealing with fractions as mathematical objects, this idea of a universal unit is quite helpful. Asking the question, which is larger, one-half or three-eighths, only makes sense if the question is one of quantity fractions. If one-half and three-eighths do not refer to a universal whole, we cannot compare them.

Although the equal-whole is a critically important concept in understanding the multiplicative structure of fractions, it appears that the equal-whole concept is difficult to acquire (Hart, 1988). Without the equal-whole concept, ordering fractions as quantities draws upon significantly different mental models of fractions compared to ordering fractions based on numeric rules.

Tasks involving comparison of quantity fractions appear to be important in identifying inappropriate part-whole thinking being applied to quantity fraction contexts. For example, when asked to determine the larger of 1/3 and 1/6, a student using different sized wholes in the comparison may be attempting to use a partition fraction to answer a quantity fraction question.

The Rational Number Project (Behr, Wachsmuth & Post, 1984; Behr, Wachsmuth, Post et al., 1984) used a range of tasks to investigate students' perception of the size of fractions. Several distinct strategies for comparing fractions were identified in their analysis of 12 Grade 4 students' responses to ordering tasks. For fractions with the same numerators, they described five distinct strategies—denominator only, numerator and denominator, reference point, manipulative and whole number dominance. The denominator only strategy, which dominated the explanations, referred only to the denominators of the fractions. For example, an explanation that referred only to the denominator, such as "the bigger the number is, the smaller the pieces get" would be recorded as denominator only. The "numerator and denominator" strategy was identified by explanations that referred to both the numerators and the denominators, indicating that the same number of parts was present but the fraction with the larger denominator had the smaller sized parts. The reference point strategy made use of a third number, such as one-half, in comparing the fractions. The manipulative strategy identified explanations that made use of pictures or manipulative materials. Whole number dominance was used to describe an ordering consistent with whole number arithmetic applied to the denominators. Some of these strategies (reference point, manipulative and whole number dominance) were also evident in comparing fractions with different numerators and denominators.

Thinking quantitatively about fractions depends on the concept images students have of fractions. Tall and Vinner (1981) describe a concept image as all of the cognitive structure in the individual's mind that is associated with a given concept, "which includes all of the mental pictures and associated properties and processes" (p.152). They refer to the evoked concept image as the portion of the concept image activated at a particular time. In this way, seemingly conflicting images may be evoked at different times without necessarily producing any sense of conflict in a child. As teaching the operations of addition and subtraction of fractions is founded on the concept of equality of fractions, understanding the evoked concept images students develop of the size of fractions is fundamental to the effective learning of rational number.

The aim of this study is to explore students' fraction concept images by examining their explanations as to which of a pair of fractions is the larger. In particular, this study addresses two questions: 1) Which strategies do Year 6 students use to compare the size of quantity fractions? 2) Do Year 6 students consistently apply the same strategy across a range of fraction comparison questions?

Method

Participants

The sample of one hundred students was chosen from two middle-class schools in the Sydney metropolitan area. The students were in the final year of primary school, Year 6. They ranged in age from 11 years 4 months to 12 years 6 months at the time the tasks were completed.

Procedure

In each class, the students were given a sheet of paper and the teacher read out seven questions. Each question asked the students to determine the larger of two quantity fractions and to explain their reason for the decision. The first three questions dealt with comparisons of common unitary quantity fractions. The fourth and fifth questions compared unitary quantity fractions where one fraction was half of the other fraction. Question six compared the complements of the unitary fractions used in question five. That is, having been asked which is bigger, one-sixth or one-third was followed by comparing five-sixths and two-thirds. The final question compared two quantity fractions where the numerator was one less than the denominator.

The questions were:

1. Which is bigger, one-half or one-third? How do you know?
2. Which is bigger, one-quarter or one-fifth? How do you know?
3. Which is bigger, one-fifth or one-sixth? How do you know?
4. Which is bigger, one-sixth or one-twelfth? How do you know?
5. Which is bigger, one-sixth or one-third? How do you know?
6. Which is bigger, two-thirds or five-sixths? How do you know?
7. Which is bigger, nine-tenths or twelve-thirteenths? How do you know?

The questions were asked at a time when the students would have completed the usual primary school instruction on fractions, which includes recording fractions in the form a/b and the equivalence of decimals, percentages and fractions but does not emphasise comparison of fractions. Ordering fractions is addressed in the high school syllabus, *Years 7–8 Mathematics Syllabus* (1990). The equal-whole is not specifically referred to in mathematics syllabus documents in NSW.

The students' responses were collected and coded in two ways: as dichotomous items (correct or incorrect) and the explanations were coded according to the method they described. That is, interpretive coding was used with the explanations. In the initial coding of the explanations, the apparent model and method were recorded. The apparent models included the area of a circle, discrete objects, area of a rectangle and methods included the use of percentages, decimals, common denominators, size of the denominator, number of cuts, size of pieces and number and size of parts. The final coding cycle looked specifically at describing the strategies implied by students' explanations.

Results and Discussion

Most students used diagrams as either part or the whole of their explanation. However, it was what they did with the diagram that was most telling. Sometimes when a child attempted to make use of an area model in his or her explanation, it became clear, as in Figure 1, that the representation did not reflect equal-area partitioning.

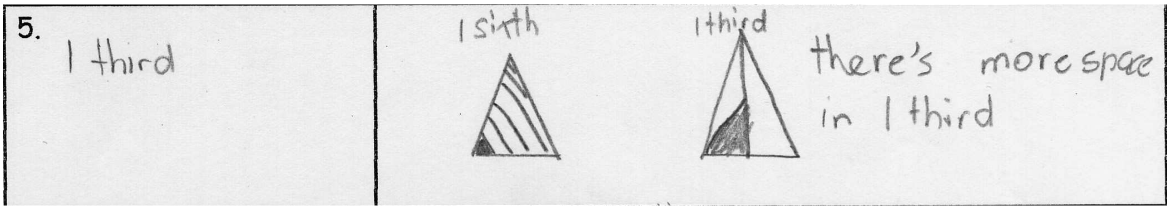

Figure 1. Which is bigger, one-third or one-sixth?

In Figure 1, it appears that the area of the pieces used to demonstrate the fractions, is not the focus of the student's attention. The number of pieces appears to define the fraction for this student, despite the appeal to the amount of space to justify the answer. Some diagrams also provided an indication that the equal-whole schema was not evident.

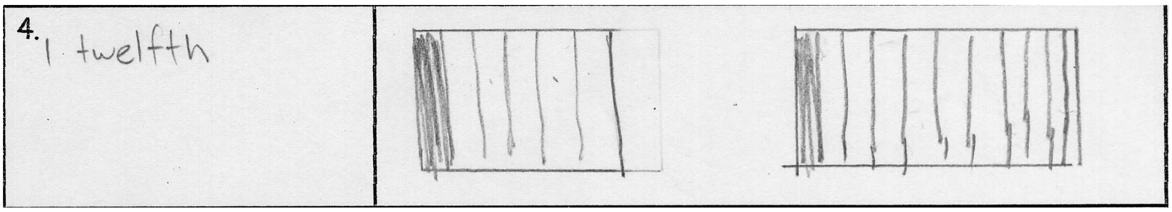

Figure 2. Which is bigger, one-sixth or one-twelfth?

The diagram (Figure 2), offered as an explanation for why one-twelfth is bigger than one-sixth, shows one part out of six shaded and for one-twelfth, a slightly smaller part is shaded out of a significantly larger unit-whole rectangle. It is conceivable that it is the increased size of the whole that becomes the basis for deciding which is the bigger fraction. This interpretation is aligned to the findings of a study by Yoshida and Kuriyama (1995) where many students drew representations in which the size of the whole each fraction represented was in direct proportion to the size of the denominator. In the current study, this was most likely to appear where students used shaded rectangles to represent the fractions. In comparing two-thirds and five-sixths it is possible to "correctly" conclude that five-sixths is larger by increasing the size of the unit whole (see Figure 3).

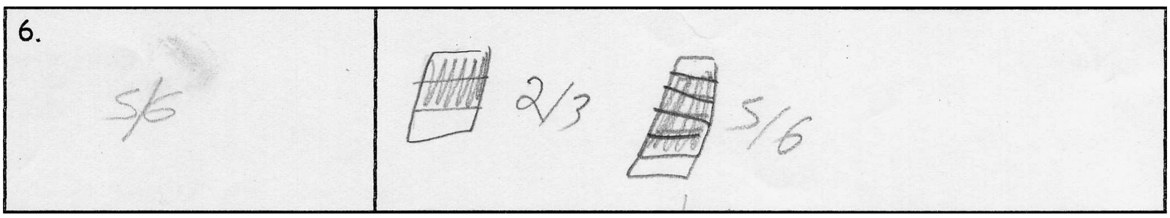

Figure 3. Five-sixths showing an increased unit-whole.

For the first five questions, students could consistently apply the rule, “the bigger the denominator the smaller the fraction” or the “greater the number of pieces the smaller the pieces”. As one student eloquently recorded in response to the third question, “…as the denominator gets bigger it gets further away from 1”. Question six and question seven necessitated a change of comparison strategy.

Table 1
Percentage Correct (dichotomous coding) N=100

Q1	Q2	Q3	Q4	Q5	Q6	Q7
96	96	92	90	95	71	53

The percentage of correct responses to the first five questions, comparing unitary fractions is very high. Only 71% correctly answered question six and this reduced to 53% for question seven. Although 41% answered all seven questions with the correct response, their responses were often achieved through spurious reasoning. For example, of those who achieved a “perfect score”, on the last question two compared the size of the numerators, one stated that 12/13 was bigger than 9/10 because it has larger numbers, two indicated that they guessed, two stated the answer as a fact, and two more used inconclusive reasoning such as multiplying both fractions by 2. Although the percentages of correct answers suggest that the majority of these students can compare fractions, many of their correct answers are based on faulty reasoning.

Questions six and seven were designed to prompt a shift in comparison strategies. Although a focus on the numeric size of the denominator, including a “bigger means smaller interpretation”, was possible with the first five questions, this strategy would be inadequate for the final two questions. These questions compared two unit fraction complements. Instead of comparing one-third and one-sixth as in question five, question six compared their complements, i.e. two-thirds and five-sixths. With question six, students could reasonably argue in terms of common denominators, converting two-thirds to four-sixths. Alternatively, they could argue proximity to 1 using the information on the relative size of one-third and one-sixth. Three students argued for their answer to question 6 based on the gap to 1 whole.

The comparison of 9/10 and 12/13 provided some fascinating insights. Two students adopted a purely additive strategy and commented that you could go from 9/10 to 12/13 by adding 3 to the top and the bottom. One concluded that 12/13 was bigger because it was 3 more (on the top and bottom) while another argued that this made the fractions the same size. In total, seven per cent of students argued that the final two fractions were equal.

The explanation that, the higher the number on the bottom the smaller the fraction, was very common. This would generally fall into the “denominator only” strategy category of Behr, Wachsmuth, Post et al. (1984). For example, one student consistently applied the explanation “if the bottom number is bigger it is smaller” to explain the responses to the first five questions. The mental model corresponding to this explanation was coded as “bigger is smaller” (BIS). This coding distinguished it from a different use of the denominator, where as the denominator got bigger the fraction was considered to be bigger. In addition to the “bigger is smaller” explanation, the student drew and labelled partitions of squares, rectangles and circles. The diagrams frequently showed area models that did not allow comparisons, as they did not depict equal wholes or equal partitioning (see Figures 4 and 5).

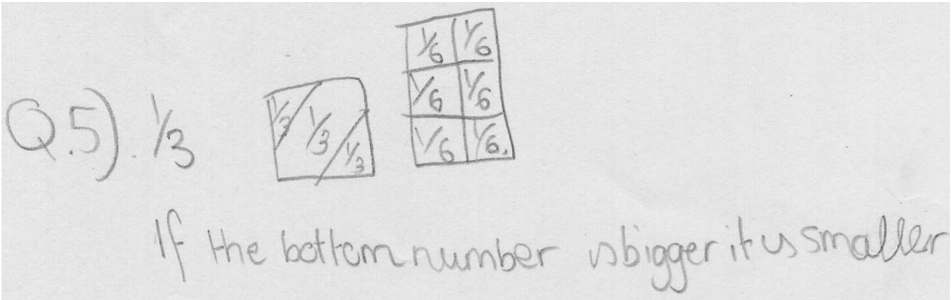

Figure 4. Neither equal-area partitioning nor equal whole schema.

The student whose responses are depicted in Figures 4 and 5 does not apply an equal whole schema nor use consistent shapes to represent the whole. In Figure 5, the student partitions a circle into five equal parts and a rectangle is partitioned into sixths to “compare” the fractions one-fifth and one-sixth.

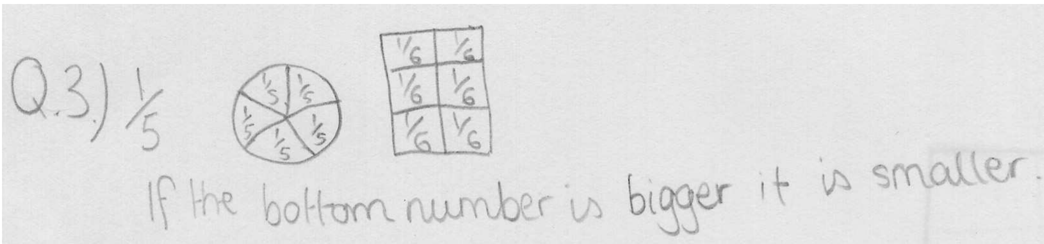

Figure 5. Pictorial comparison of fifths and sixths in different shapes.

A focus on the whole number denominator meant that some students interpreted the size of the denominator as being proportional to the size of the fraction (see Figure 6). Behr, Wachsmuth, Post et al. (1984) would have described this thinking as “whole number dominance” even though they would have assigned this response to the category of “manipulatives”, as the child explained his or her response using pictures.

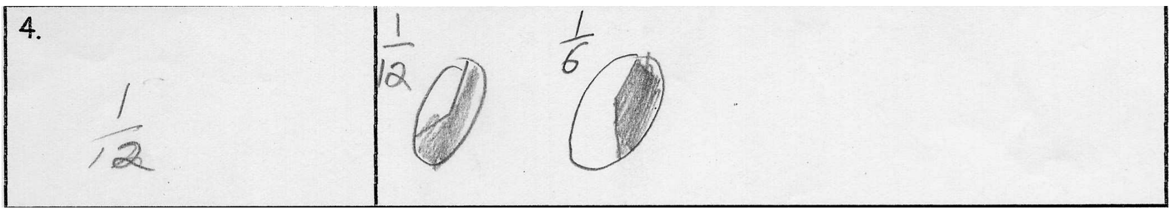

Figure 6. Fractions proportional to whole number denominators.

The methods the Year 6 students used to compare fractions were quite stable across questions with 48% of students consistently applying the same strategy used with the unitary fractions to the comparison of non-unitary fractions.

Table 2
Strategies used to Compare the Sizes of Unitary and Non-unitary Fractions

Strategy	Unitary (Q1)	Non-unitary (Q7)
Area model	44	34
Number and size of parts	10	0
Bigger denominator is smaller fraction (BIS)	6	12
Parts of a common whole (e.g., 3, 6, 9, 100)	7	5
Additive reasoning (9+3)/(10+3) = 12/13	0	6
Conversion procedure (Common Denominator)	4	10
Conversion procedure (Decimal)	4	0
Conversion procedure (Percentage)	3	3
Number of parts	4	4
Size of complement to 1	0	4
Other (e.g. fact, changed question, unclear, guessed, no explanation)	18	22

Some of the strategies used by students and outlined in Table 2 were specific to the type of question. For example, using the size of the gap to one-whole was only sensible in the final two questions. However, some methods persisted even when they were difficult to apply. By question 7, comparing $\frac{9}{10}$ and $\frac{12}{13}$, 34 of the Year 6 students continued to use comparison of areas to determine the larger fraction.

Conclusion

This study indicated that the strategies that students use to compare the size of fractions are diverse. Some correspond to taught procedures, such as converting to common denominators or percentages. Others appear to be invented strategies that may reflect the learner's struggle to make sense of a non-transparent symbol system within a complex concept image. The number of "correct" answers students obtained through incorrect reasoning was remarkably high. The reasoning provided by students in justifying their answers proved a much richer insight into their concept images than the total correct.

Drawings representing the area model dominated the Year 6 students' explanations. Indeed, the comparison of area to determine the larger of two fractions persisted even when this strategy became extremely cumbersome, as in comparing nine-tenths or twelve-thirteenths.

Students appear to use a strategy that encapsulates the inverse relationship between the magnitude of the whole number within the denominator and the size of the resulting fraction to compare the size of fractions. Six students specifically gave this as an explanation in response to Question 1 while 10 more referred to the related idea of the number and size of the parts. Twelve students used this mode of reasoning for their answer to the final question.

What is particularly telling from this study is the number of students who concluded that nine-tenths and twelve-thirteenths were the same. An answer that they are the same is clearly a reasoned response as it goes beyond the question frame. Seven students argued that nine-tenths and twelve-thirteenths were equal, one argued equality using an area model. As the algorithmic approach to addition and subtraction of fractions is founded on the notion of equivalent fractions, care needs to be taken in inferring students' understanding of the equality of fractions from the number of correct answers on standard assessments.

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