Subject Matter Knowledge: Mathematical Errors and Misconceptions of Beginning Pre-Service Teachers

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This study describes errors and misconceptions of pre-service primary teachers, at course entry, across the mathematics curriculum. A Rasch analysis of a multiple-choice mathematics test uncovered patterns of errors in a cohort of 426 students at the beginning of their teacher education course. These errors were of varying sophistication. A map of an individual's errors is also presented and we discuss how teacher educators and students can confront subject matter knowledge misconceptions using the diagnostic capability of the test.

Knowledge of the common mathematical errors and misconceptions of *children* can provide teachers with an insight into student thinking and a focus for teaching and learning (Bell et al, 1985; Black & Wiliam, 1998; Hart, 1981; Schmidt et al., 1996; Stigler & Hiebert, 1999; Williams & Ryan, 2000). A social constructivist view of learning suggests that errors are ripe for classroom consideration; via discussion, justification, persuasion and finally even change of mind, so that it is the student who reorganises their own conception (Cobb & Bauersfeld, 1995; Cobb, Yackel & McClain, 2000; Ryan & Williams, 2003; Tsamir & Tirosh, 2003).

Shulman's (1986) three categories of teacher content knowledge – subject matter content knowledge, pedagogical content knowledge and curricular knowledge – are intertwined in practice. Pedagogical content knowledge includes

> an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them … If those preconceptions are misconceptions, which they often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners. (Shulman, 1986, pp. 9-10)

Subject matter knowledge is more than knowledge of facts or concepts – it requires knowledge of both the substantive structure (facts and their organising principles) and syntactic structure (legitimacy principles for the rules) of a subject domain. The transformation of subject matter knowledge into pedagogical content knowledge is a significant focus in teacher education (Goulding, Rowland & Barber, 2002). We suggest that pre-service teachers who confront their *own* mathematical errors, misconceptions and strategies in order to reorganise their subject matter knowledge, have an opportunity to develop a rich pedagogical content knowledge.

Most teacher education institutions require a minimum level of school mathematics achievement in their admissions procedures for applicants (typically in Victoria it is a year 11 mathematics). However, such levels do not provide fine detail about subject matter knowledge. Our research attempts to uncover more detail about beginning pre-service teacher subject knowledge in mathematics, including current attainment, patterns of errors (behaviours) and misconceptions (inferred cognitive structures), and makes suggestions for diagnostic teaching in pre-service teacher education courses.

It is mandatory in the United Kingdom for teacher education courses to provide evidence of secure subject knowledge of students during their training (TTA, 2003) and all beginning teachers must pass a numeracy test (as well as literacy and information and communication technology tests) to gain qualified teacher status by the end of their induction period. Some states in the USA use professional assessments of reading, mathematics and writing for beginning teachers as part of their teacher licensure process (Gitmer, Latham & Ziomek, 1999; Study Guide for the Pre-professional Skills Test, 2003). However, a search of the literature found no numeracy test for pre-service teachers that supplied detail of mathematical errors and misconceptions across the mathematics curriculum. Our work provides the missing diagnostic information: what are the misconceptions, what are the ability levels of students that hold them, and how can the errors be used so that pre-service teachers can re-organise their mathematical understanding?

A Teacher Mathematics Test

The *ACER Teacher Education Mathematics Test* (TEMT) (ACER, 2004) is designed to test the mathematical attainment of beginning primary trainee teachers[1] *and* to uncover errors, misconceptions and strategies in order to provide diagnostic feedback. A 'primary teacher curriculum' was first constructed from a consideration of the Victorian *Curriculum and Standards Framework* (*CSF*) (Board of Studies, 1995; 2000), *Mathematics – a Curriculum Profile for Australian Schools* (Curriculum Corporation, 1994) and the UK *Initial Teacher Training National Curriculum* (Department for Education and Employment, 1998; Teacher Training Agency [TTA], 2003). TEMT assumes level 5/6 attainment on the *CSF*[2]. Multiple-choice items were written to test both substantive and syntactic knowledge of the primary teacher curriculum.

The TEMT test items were written with diagnostic coding for most distracters (three or four per item). This paper reports on some of the errors and misconceptions uncovered by the TEMT. A range of mathematics education research on children's and teachers' knowledge and errors informed the writing of the TEMT items and choice of distracters (e.g., Ashlock, 2002; Coben, 2003; Hart, 1981; Ma, 1999; Ni, 2000; Pitkethly & Hunting, 1996; Rowland, Heal, Barber & Martyn, 1998; Ryan & Williams, 2000; Thompson & Saldanha, 2003; Williams & Ryan, 2000). It was also seen to be important to provide adult contexts for test items and to take advantage of the presumed higher reading ability of adult students.

Methodology

Students across three different degree courses (total N = 426) took a TEMT test in the first few weeks of the first year of their teacher education degree at a university in Victoria in 2004. There were three equivalent forms of the test with 15 link items, with each test containing 45 multiple-choice items (105 items in total). The use of calculators was not allowed. The tests were timed for a 45-minute testing period. The six curriculum strands covered were: Number (16 items in each test), Measurement (8), Space and Shape (8), Chance and Data (6), Algebra (5), and Reasoning and Proof (2). Marks were not deducted for incorrect responses.

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[1] It can also be used as a tool to aid selection into tertiary courses.

[2] According to the 2000 edition of the *CSF* (*CSF II*), "It is expected that the majority of students will be demonstrating achievement at level 6 by the end of Year 10 – a realistic requirement for functional numeracy."

The three test forms were found to be well-equated. The test was used to report a total *achievement score* to the university which reported that the test scores generally correlated well with the trainees' grades, some months later, on the Basic Skills Test that the university had been using for some years. Sub-scores on each strand were also reported.

Analysis

A Rasch analysis (Rasch, 1980; Wright & Stone, 1979) was undertaken using Quest software (Adams & Khoo, 1996). Our data were found to be compatible with the Rasch model. Test reliability and goodness of fit are reported in Ryan and McCrae (2005). Quest provides classical statistics as well as *item estimates* (item difficulty estimates with the mean difficulty set at zero), *case estimates* (student ability) and fit statistics.

An *item map* output uses a logit scale (usually from –3 to 3) on which both items and cases are calibrated. A student with an ability estimate of, say, 1.0 is likely (probability level of 0.5) to have correctly answered all items having difficulty below the same estimate (here, 1.0).

An *item analysis* output provides, amongst other statistics, the frequency of each response (correct and incorrect) and a *mean ability estimate* of the students making each response (correct and each incorrect). It is therefore possible to consider which students are making which errors and to consider the 'sophistication' of each incorrect response.

For example, an item testing the skill of multiplication of decimals "$0.3 \times 0.24 =$ " showed a range of errors (see Table 1).

Table 1

Item Analysis for Multiplication of Decimals

Response	**Inferred Misconception**	**Frequency**	**Mean Ability** (logit)
A. 0.072	CORRECT	36.1%	1.35
B. 0.08	0.3 is one-third or decimal implies division	3.5%	0.76
C. 0.72	3×24 and adjust to 2 decimal places	41.1%	0.14
D. 0.8	0.3 is one-third or a decimal implies division and adjust to 1 decimal place	2.8%	-0.52
E. 7.2	$0.3 \times 0.24 = 3 \times 2.4$	15.3%	0.84
Omitted		1.4%	-2.29

Table 1 shows that the mean ability of students incorrectly selecting option C was 0.14, while those students who incorrectly selected option E had a higher mean ability of 0.84. E appears to be a more 'sophisticated' error than C (and the other distracters in this item). By this means, we can locate all errors in terms of ability estimates of students.

Another item "Write $912 + \frac{4}{100}$ in decimal form" indicated a significant percentage (a total of about 24%) of students with various misconceptions of place value (see Table 2).

A further item "$300.62 \div 100$", that tested division by 100, showed a range of errors again relating to place value understanding (see Table 3). Option E was the most common error and the mean ability of students making this error (0.10) was higher than for the other distracters. Students who chose option E apparently separated ‘whole’ and ‘decimal’ as different entities.

Table 2

Item Analysis: Place Value Understanding

Response	Inferred Misconception	Frequency	Mean Ability (logit)
A. 912.4	‘hundredths’ is first decimal place	3.5%	-0.13
B. 912.04	CORRECT	76.3%	0.91
C. 912.004	‘Unit-ths, tenths, hundredths’	12.2%	-0.30
D. 912.25	4/100 is ¼ or 100 divided by 4 gives the decimal or 1/25 is 0.25	6.0%	-0.10
E. 912.025	100 divided by 4 is 25 and ‘unit-ths, tenths, hundredths’	1.6%	-0.10
Omitted		0.7%	-0.77

This ‘separation’ strategy (see Table 3) was also evident in other items where the operation was multiplication and also where the number was mixed (integer and fraction). The misconception underlying the strategy is important because it also appears to be at the root of the well-documented ‘decimal point ignored’ and the ‘longest/largest decimal is smallest’ errors (Assessment of Performance Unit [APU], 1982).

Table 3

Item Analysis: Division by 100

Response	Inferred Misconception	Frequency	Mean Ability (logit)
A. 30062	‘Move’ decimal point 2 places to the right	0%	
B. 30.062	‘move’ decimal 1 place to the left	6.4%	-0.13
C. 30.62	‘cancel’ a zero	2.6%	-0.18
D. 3.0062	CORRECT	68.8%	0.98
E. 3.62	Integer-decimal separation or ‘cancel 2 zeros’	22.0%	0.10
Omitted		0%	

We have reported above on some Number items to demonstrate a *pattern* of place value (mis)conceptions across items. Other Number items indicated that many students had faulty algorithms and often operated in a one-step process only. Items involving two-step operations were considerably more difficult for the students. There are similar patterns for other strands of the curriculum. For example, in Measurement the reading of linear scales in various contexts indicated that many students were counting the ‘tick marks’ rather than the ‘gaps’ and not accounting for non-unitary scales. There were also errors in spatial vocabulary (perpendicular/diagonal/hypotenuse confusion; similar/congruent confusion) and measurement vocabulary (area/perimeter confusion).

Altogether 44% of the items contained at least one distracter that, from the literature, is believed to be an error that diagnoses a significant misconception. Ninety-three percent of these errors occurred significantly more than would be expected from students' guessing and hence is prima facie evidence of the misconception.

Individual Profile: Kidmap

The Quest item analysis statistics were used to identify patterns of errors for the cohort of students. Quest also produces a *kidmap* that is an output for each individual identifying their correct *and* error response patterns.

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--------------------------------------------- K I D M A P----------------------------------------
Student: 6                                              ability:    0.90
group:   all                                            fit:        1.10
scale:   all                                            % score:  64.44
--------------Harder Achieved ------------------------------Harder Not Achieved -------------
                                    |   |
                                  truncated
                                25  |   |
                                    |   |
                                    |   |
                                    |   |
                                    |   |
                                    |   |
                                    |   |   28(3)
                                    |   |   13(1)
                                    |   |
                                42  |   |   39(1)
                                    |   |   27(3)   41(2)   44(1)
.........................................    5(1)   32(5)
                                12  |   |
                                45  |   |
                                    |XXX|    8(2)
                                40  |   |   33(4)
                        38      30    ..................................
                        23      15  |   |   35(5)
                        37      29  |   |
                                    |   |    6(3)
                                    |   |
                14       4       3  |   |   18(3)
                                 7  |   |
                                    |   |
        22      17       2       1  |   |    9(1)
43      34      26      19      11  |   |
                        24      10  |   |   16(2)
                                    |   |
                                36  |   |
                                31  |   |   21(1)
                                    |   |
                                    |   |
                                20  |   |
                                  truncated
                                    |   |
-------------Easier Achieved ------------------------------Easier Not Achieved -----------
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Figure 1. Kidmap for student 6: Pattern of response.

An example is shown in Figure 1 where student 6 has an ability estimate of 0.90, a mean square infit statistic of 1.10 and a total score of 64.44%. The row of Xs (centre of the map) indicates the ability estimate of the student (0.90 in this case) and the dotted lines are drawn at ± 1 standard error.

The items are plotted at their difficulty level in logits. The items *not* achieved by the student are plotted on the right-hand side of the map. The actual response made for each incorrect item is indicated in parentheses: for example, student 6 would have been expected to have achieved item 35 (below the lower dotted line) but responded incorrectly with option 5. This enables the diagnostic errors indicated in the bottom right quadrant to be studied more carefully.

For example, we analyse the case of student 6 in order to demonstrate how the *kidmap* profile can be used on an individual basis. In the right-hand bottom corner (4th quadrant) of student 6's *kidmap* (Figure 3), there are eight 'easier not achieved' items that the student was expected ($p \geq 0.5$) to have achieved given his/her ability estimate. These errors are examined in detail with an inferred misconception for the choice of distracter in Table 4.

Table 4

Analysis of Student 6's 'Easier not Achieved' Items

Item	**Difficulty** (logit)	**Description**	**Inferred Misconception**
8(2)	0.95	Algebra: general statements	Variable as specific number
33(4)	0.68	Number: Identifying ratio within several ratios	Additive tendency
35(5)	0.43	Number: Calculating surface area	Area/volume confusion
6(3)	0.11	Space: Cartesian co-ordinates	Co-ordinate reversal
9(1)	-0.69	Reasoning: logic	Triangle prototype (equal angles)
18(3)	-0.97	Measurement: grams to kilograms	100g is 1 kg
16(2)	-1.09	Number: Fraction representation	Unequal parts of whole treated as equal
21(1)	-1.56	Algebra: words to symbols	'more than' implies multiply

These errors are particularly interesting because student 6 was expected to have responded correctly for these items – they might indicate gaps or 'bugs' in knowledge. The items in the top left corner (1st quadrant) are the 'harder achieved' items. The correct responses here may suggest guessing in the multiple-choice test format.

Conclusion

We have shown that it is possible to construct an instrument designed for the measurement of teachers' subject knowledge that also has diagnostic properties, by selecting and calibrating items that have diagnostic potential (mainly from the literature on children's misconceptions) in the test construction process. Many items revealed that significant proportions of a cohort on entry to initial teacher education have the targeted errors/misconceptions. It was further shown that a student's *kidmap* can be used as a tool for identifying an individual student's profile of errors, hence providing automated feedback of potential diagnostic value to the student.

Implications for Initial Teacher Education

Errors uncovered by the *ACER TEMT* could form the basis of group discussion; considering why the given reasoning is correct or incorrect, what warrant is presented to support a claim, and what mathematical 'tools' or artefacts are called on to demonstrate or help to re-organise understanding. This focus could be of value to a beginning teacher and to the tertiary educator seeking to gain insight into students' misunderstandings. A teacher educator could use cohort patterns as the basis for conflict peer group discussion of different conceptions (Bell et al, 1985; Cobb & Bauersfeld, 1995; Ryan & Williams, 2000) to support pre-service teacher learning. Within group discussion, students can be asked to listen to others via discussion, justification, persuasion and finally even change of mind, so that it is the student who reorganises their own conception. Toulmin's (1958) model of argument is helpful here and a range of errors is valuable in such conflict discussion.

For example, the separation strategy (indicated by "300.62 ÷ 100 = 3.62" in Table 3) is suitable for such discussion where the meaning of number and division are paramount. What representations do different students draw on to justify their claims? Which representations are successful in shifting or strengthening a conception? For a pre-service teacher, it is the use of *representations* that may shift procedural behaviour towards conceptual understanding. Indeed representations are the life-blood of teaching and the basis of pedagogical teacher knowledge.

A *kidmap* provides a profile of an individual's errors. We suggest that pre-service teachers could use their *kidmaps* to investigate their own understandings. We are currently researching how pre-service teachers can do this successfully. Teacher errors deserve attention not least to avoid transfer to children in schools. Errors provide opportunities for pre-service teachers to examine the basis for their own understandings, as well as identifying areas for attention by teacher educators.

Pedagogical content knowledge is characterised as including '"the most useful forms of representation of … ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p.9). We believe that the beginning teacher needs to first make the subject comprehensible to him/herself – to examine the "veritable armamentarium of alternative forms of representation" (Shulman, 1986, p.9) so that mathematics learning is modelled dynamically as change, re-organisation and confirmation of ideas.

References

Adams, R. J., & Khoo, S-T. (1996). *ACER Quest: The interactive test analysis system*. Melbourne: ACER.

APU (Assessment of Performance Unit). (1982). *A review of monitoring in mathematics 1978-1982.* London: DES.

Ashlock, R. B. (2002). *Error patterns in computation: Using error patterns to improve instruction* (Eighth edition). Upper Saddle River, NJ: Merrill Prentice Hall.

Bell, A. W., Swan, M., Onslow, B., Pratt, K., & Purdy, D. (1985). *Diagnostic teaching: Teaching for lifelong learning.* Nottingham: University of Nottingham, Shell Centre for Mathematics Education.

Black, P., & Wiliam, D. (1998). *Inside the black box: Raising standards through classroom assessment.* .London: Department of Education and Professional Studies, King's College.

Board of Studies. (1995). *Curriculum and standards framework: Mathematics.* Melbourne: Board of Studies.

Board of Studies. (2000). *Curriculum and standards framework II: Mathematics* (Revised edition). Melbourne: Board of Studies.

Cobb P., & Bauersfeld, H. (1995). (Eds.) *The emergence of mathematical meaning: Interaction in classroom cultures.* Hillsdale, NJ: Erlbaum.

Cobb P. E., Yackel, E., & McClain, K. (Eds.) (2000*). Symbolizing and communicating in mathematics classrooms*. Mahwah, NJ: Lawrence Erlbaum.
Coben, D. (2003). *Adult numeracy: Review of research and related literature. Research review.* Sydney: National Research and Development Centre for Adult Literacy and Numeracy.
Curriculum Corporation. (1994). *Mathematics – A curriculum profile for Australian schools.* Melbourne: Author. Retrieved from http://www.nrdc.org.uk/uploads/documents/doc_2802.pdf
Department for Education and Employment. (1998). *Teaching, high status, high standards: Circular 4/98* [Annex D]. London: Her Majesty's Stationery Office.
Gitmer, D. H., Latham, A. S., & Ziomek, R. (1999). *The academic quality of prospective teachers: The impact of admissions and licensure testing.* London: Teaching and Learning Research Report. Educational Testing Service.
Goulding, M., Rowland, T., & Barber, P. (2002). Does it matter? Primary teacher trainees' subject knowledge in mathematics. *British Educational Research Journal, 28*(5), 689-704.
Hart, K. (1981). *Children's understanding of mathematics 11–16*. London: John Murray.
Ma, Liping (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.
Ni, Y. (2000). How valid is it to use number lines to measure children's conceptual knowledge about rational number? *Educational Psychology, 20*(2), 139-152.
Pitkethly, A., & Hunting, R. (1996) A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics*, *30*(1), 5-38.
Rasch, G (1980). *Probabilistic models for some intelligence and attainment tests.* Chicago: The University of Chicago Press.
Rowland, T., Heal, C., Barber, P., &. Martyn, S. (1998). Mind the 'gaps': primary teacher trainees' mathematics subject knowledge. In E. Bills (Ed.), *Proceedings of the British Society for Research in Learning Mathematics Conference* (pp. 91-96). Coventry: University of Warwick.
Ryan, J., & McCrae, B. (2005, accepted). Pre-service teachers' mathematics subject knowledge: Admissions testing and learning profiles (Short Oral Communication). *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education.* Melbourne, Australia: PME.
Ryan, J., & Williams, J. (2000). *Mathematical discussions with children: Exploring methods and misconceptions as a teaching strategy.* Manchester: Centre for Mathematics Education: University of Manchester.
Schmidt, W. H., Jorde, D., Cogan, L. S., Barrier, E., Gonzalo, I., Moser, U., Shimuzu, K., Sawada, T., Valverde, G. A., McKnight, C., Prawat, R. S., Wiley, D. E., Raizen, S. A., Britton, E. D., & Wolfe, R. G. (1996). *Characterising pedagogical flow: An investigation of mathematics and science teaching in six countries.* Dordrecht: Kluwer.
Shulman, L. (1986). Those who understand: knowledge growth in teachers. *Educational Researcher, 15*(2), 4-14.
Stigler, J.W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press (Simon & Schuster Inc.).
Study guide for the pre-professional skills tests (PPST). (2003). (Second edition.) Princeton NJ: Educational Testing Service. PRAXIS series.
Teacher Training Agency (2003). *Qualifying to teach: Professional standards for qualified teacher status and requirements for initial teacher training*. London: Teacher Training Agency.
Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to Principles and Standards for school* mathematics (pp. 95-113). Reston, VA: NCTM.
Toulmin, S. (1958). *The uses of argument.* Cambridge: Cambridge University Press.
Tsamir, P., & Tirosh, D. (2003). *Errors in an in-service mathematics teacher classroom: What do we know about errors in classrooms?* Paper presented at the 7th International Symposium of Elementary Mathematics Teaching. Prague, Czechoslovakia.
Williams, J., & Ryan, J. (2000). National testing and the improvement of classroom teaching: Can they coexist? *British Educational Research Journal, 26*(1), 49-73.
Wright, B., & Stone, M. H. (1979). *Best test design: Rasch measurement.* Chicago: MESA Press.