Patterns Supporting the Development of Early Algebraic Thinking

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This paper examines teacher actions that support young children to consider repeating pattens as co-variational (functional) relationships, to use this understanding to predict uncountable steps in the relationships, to express these relationships in general terms, and use repeating patterns to introduce proportional thinking. A teaching experiment was conducted in two classrooms, comprising of a total of 45 children whose average age was 9 years and 6 months. This experiment focused on exploring teacher actions (including the use of concrete materials, recording of data, and questions asked) that supported young children's development of co-variational reasoning. The results indicated that explicit instruction assisted children to find patterns across the table as well as down the table, to find the relationships between the number of tiles and an uncountable number of repeats. Also the results indicate that young children are capable of not only thinking about the relationship between two data sets, but also of expressing this relationship in a very abstract form.

Mathematics has been referred to as the Science of patterns (Steen, 1990). Abstracting patterns is the basis of structural knowledge, the goal of mathematics learning in the research literature (Johnassen, Beissner & Yacci, 1993; Sfard, 1991). Thus the focus of mathematics teaching should be directed to fostering fundamental skills in generalising, and expressing and systematically justifying generalisations (Kaput & Blanton, 2001). Such experiences give rise to understandings that are independent of the numbers or objects being operated on (e.g., a+b = b+a regardless of whether a and b are whole numbers, decimals, or variables). Ohlsson (1993) names such understanding abstract schema and argues they are more likely to promote transfer to other mathematical notions than a schema based on particular numbers or content.

This belief is also reflected in recent international and national syllabuses (e.g. Queensland Studies Authority, 2005; National Council for Teaching Mathematics, 2000) where Patterns and Algebra are now themes starting at the early years. Yet, as reported by Waters (2004), there appears to be very limited literature on patterning per se, and particularly on generalising patterns and expressing and justifying these generalisations. Most past studies have used patterning ability as an indicator of readiness for other mathematical ideas or as a precursor to reasoning (e.g. English, 2004; Klein & Starkey, 2003).

A common activity that occurs in many early years' classrooms in the Australian context is the exploration of simple repeating and growing patterns using shapes, colours, movement, feel and sound. Typically young children are asked to copy and continue these patterns, identify the repeating or growing part, and find missing elements; a focus on single variational thinking where the variation occurs within the pattern itself (e.g., what comes next). Approaches for introducing algebra to young adolescents (12-13 years) build on early explorations of visual patterns, using the patterns to generate algebraic expressions (Bennett, 1988), with a focus on functional thinking, and thinking between two data sets (e.g. comparing elements of a pattern to their position in the pattern). Past research has indicated that many young adolescents experience difficulties with the transition to patterns as functions (Redden, 1996; Stacey & MacGregor, 1995; Warren, 1996, 2000). These difficulties include the lack of appropriate language needed to describe this relationship, the propensity to use an additive strategy for describing generalisations (i.e., a focus on a single data set rather than the relationship between two data sets), and an inability to visualise spatially or complete patterns. However, young children are believed to be capable of thinking functionally at an early age (Blanton & Kaput, 2004), that is how values are changed or mapped to other quantities, commonly referred to in the literature as co-variational thinking (Chazan, 1996).

This paper investigates instruction that assists young children generalise and formalise their mathematical thinking, and come to some understanding of situations involving repeating patterns. Two lessons were designed to extend children's thinking about repeating patterns to include variation between the elements of the pattern and the number of repeats. The specific aims of the investigation were to: (a) document the implementation of the lesson; (b) identify examples of children's algebraic and functional thinking; and (c) determine teacher actions, children's material use and classroom activities that begin to facilitate mathematical thinking

Method

The methodology adopted was that of a Teaching Experiment, the conjecture driven approach of Confrey and Lachance (2000) and was exploratory in nature. Two lessons were conducted in two Year 4 classrooms from one low and one middle socio-economic elementary schools from an inner city suburb of a major city. The sample, therefore, comprised 45 students (average age of 9 years and 6 months), two classroom teachers (Amy and Sarah) and 2 researchers. The lessons reported in this paper were those conducted by one of the researchers (teacher/researcher). During and in between each lesson hypotheses were conceived 'on the fly' (Steffe & Thompson, 2000) and were responsive to the teacher-researcher and the students. The two classes had shown differing levels of ability: This decision was based on (a) previous teaching experiments conducted in these classes (Warren & Cooper, 2003), (b) the beliefs of the two classroom teachers, and (c) students' records of achievement. Amy's class (24 students) was of average ability, whereas Sarah's class (21 students) was of average to high ability. The lessons occurred on consecutive days, starting each day with a lesson in Amy's class and finishing with the revised lesson in Sarah's class.

The lessons were of approximately one hour's duration. The first lesson consisted of four phases, namely, (1) copying and continuing a simple ABBABBABBABB pattern (represented with red and green tiles); (2) uncovering progressive sets of repeats, counting the number of A's and B's in these sets, and recording the data in a table; (3) identifying relationships within the table; and, (4) using this relationship to predict the number of A's, B's and total tiles in an uncountable number of repeats. The materials used were red and green square tiles. The second lesson focussed on extending these understandings to include more complex repeating patterns, expressing the co-variational relationships in general terms and as ratios and rates of change.

During the teaching phases, another researcher and classroom teacher acted as participant observers, recording field notes of significant events including student-teacher/researcher interactions. Both lessons were videotaped using two video cameras, one on the teacher and another on the students, particularly focussing on the students that actively participated in the discussion. Children were also encouraged to record their thinking throughout each phase of the lessons. Every attempt was made to ensure that the recordings were indeed that particular child's thinking by allowing no erasers, emphasising that we were interested in their thinking rather than correct answers, and collecting the data at regular intervals throughout the lessons. At the completion of the teaching phase, the researcher and teacher reflected on their field notes, endeavouring to minimise the distortions inherent in this form of data collection, and arrive at a common perspective of the instruction that occurred and the thinking exhibited by the children participating in the classroom discussions. The video-tapes were transcribed.

In order to ascertain children's capabilities with repeating and growing patterns before the teaching phase, a pre-test was administrated. Figure 1 presents the questions asked in the pre-test.

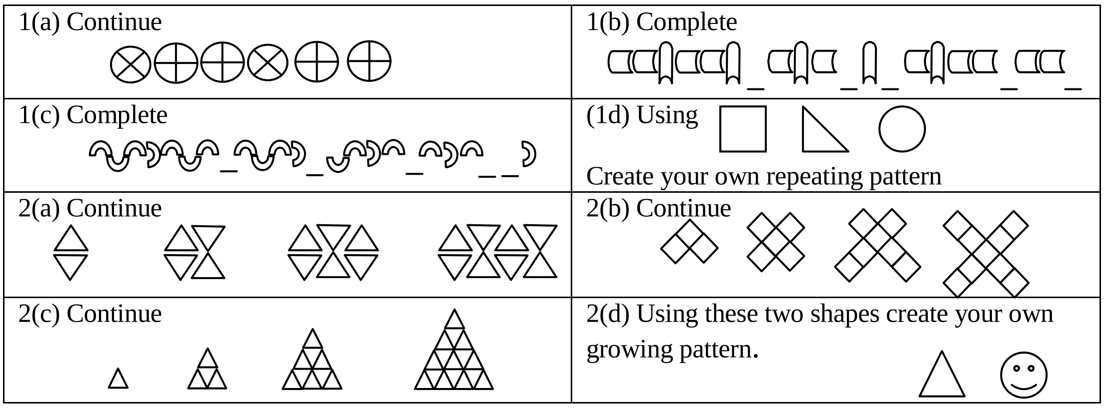


Figure 1. Repeating and growing pattern questions.

Results

The pre test (see Figure 1) was administered by the classroom teacher. The frequency of responses for the 2 questions are presented in Table 1.

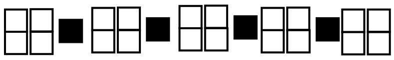
Table 1
Frequency of Responses to the Repeating and Growing Pattern Question

	Repeating patterns				Growing patterns			
	1a	1b	1c	1d	2a	2b	2c	2d
Incorrect	3		13	4	16	21	21	24
Correct	42	45	32	39	29	22	21	19
No answer				2			1	1

There was a significant difference between results of the repeating pattern component of the test and the growing pattern component, and also between the results from the two classes. It was conjectured that these children had had many more prior experiences with repeating patterns than growing patterns and they had a firm understanding of the developmental phases involved in understanding repeating patterns, namely, continuing, completing and creating. There was also significant differences between the two classes for both types of patterns (Repeating $F_{1,44}$=9.580; Growing $F_{1,44}$=24.040: p<.05), with Sarah's class exhibiting a greater ability for both. Thus the teaching phases focussed on extending these understandings to include repeating patterns as co-variations between data sets.

The two lessons in each classroom were compared in terms of differences and similarities between teaching actions and student responses. Conclusions were drawn with respect to the relative effectiveness of the teaching and the form and nature of any development of algebraic thinking. Due to space constraints, only the main conclusions drawn from the data are discussed in this paper.

Lesson 1: Classroom 1 – Amy's class

Phase 1: The repeating of GGRGGRGGRGGRGG (where R=red tile and G=green tile) was successfully created by all students. In order to ascertain if they could translate between repeating patterns, they were asked to 'use the tiles to make a new repeating pattern that is the same as this pattern'. Some common responses were RRGRRGRR, with a typical explanation being *instead of starting with 2 greens you start with 2 reds*. Another common response was  with the explanation being *same same different same same different.* One child (Sam) made the pattern RGGRGGRGGRGG and when asked if this pattern was a new pattern as compared with the first one, he responded, *it is different because it starts with R instead of GG,* suggesting that perhaps the starting position is seen as an important characteristic of repeating pattern. This conjecture was tested by asking who thought that GGRGGRGGRGGR was the same pattern as or different from RGGRGGRGGRGG. Many believed that they were different, with Sally saying that they were different because *it is different same same, different same same, not same same different*. But some thought that it was the same pattern *we just have a different start.*

Phase 2: Reforming the repeating pattern GGRGGRGGR, children were asked to use a card to expose the first set of the repeating pattern and record the number of green tiles and red tiles, comprising two columns. This process was reiterated for 2 sets, 3 sets, 4 sets and 5 sets. Most could successfully complete the table but some had to physically count the tiles each time. Figure 2 illustrates the steps in the task.

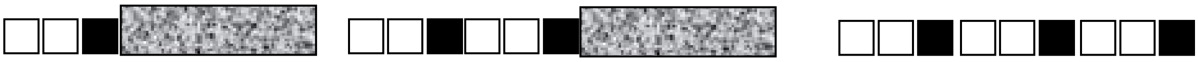
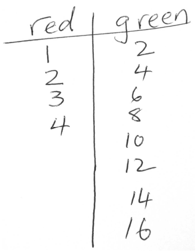


Figure 2. Uncovering the GGRGGRGGR repeating pattern.

When asked to extend this table for other repeat numbers, many children first extended one side of the table downwards and then completed the other. The following photograph illustrates this trend.



It seemed that many children extended the table by patterning down the table instead of across it. This conjecture is confirmed by the following conversations. 'Who wants to explain the pattern to me?' Annabelle said, *You keep adding 2 to the reds and 1 to the greens*. But as Steven indicated not all children thought like this. His response to this question was, *The reds are double the greens*. "So what if I had 5 greens how many reds?" Steven said, *10*.

Phase 3: In order to ascertain if these children could correctly predict the number of tiles for an uncountable number of repeats, they were asked, "What if I had 212 red tiles, how many green tiles would I have?" Annabelle responded, *214 – because it is two more than the reds*. It was conjectured that the focus of her thinking was on adding 2 to the reds rather than the relationship between the number of red tiles to the number of green tiles. Steven's response was, *No. You count the reds and you count the greens and one red is worth 2 greens so it is 2 lots of 212*. Children were then asked to complete a similar question on the worksheet where the pattern was 

Phase 4: Responses to the worksheet question were shared. Most children had correctly completed the table for 4 repeats, that is, entering the ordered pairs (3,1), (6,2), (9,3) and (12,4) in the table under the headings hearts and squares. They were then asked, "If I had

100 repeats, how many squares would I have and how many hearts would I have?” Kyla said, *100 squares and 108 hearts.* When asked, “How did you work it out?” Klya pointed to the last entry in her table and said *there are 8 more hearts than rectangles so it is 108.* In this instance she was using one example to reach her generalisation. When the class was asked if she was correct, many agreed with her. Cameron, who disagreed, pointed to the table and said, *There are 300 hearts and 100 rectangles because* (pointing to the entries in the table) *3 times 1 is 3, 3 times 2 is 6, 3 times 3 is 9.* In further conversations with the children it appeared that many did not know their three times tables. This was confirmed in the transcripts and the field notes taken by the second researcher.

As a result of the difficulties these children experienced with reaching generalisations, it was decided to change the lesson so that (a) the recording in the table included the number of repeats, and (b) processes were developed that directed children to look for patterns across the table (co-variational thinking) as well as down the table (single variational thinking).

Classroom 2 – Sarah’s class

Phase 1: This phase attempted to probe more deeply the children’s understanding of repeating patterns. The following patterns were drawn on the board and they were asked to copy and continue these patterns with their tiles

(a) ■■□■■□ (b) □■■■□■

For the first pattern, all of the children in this class could copy and continue the pattern. When asked, “What part repeats?” Jill said, *Red Red.* “What else repeats?” Brian said, *The green. The green comes after the red.* For the second pattern two different responses were given by these children. Most continued the pattern by simply adding ■■□ stating the repeating part was GRRR. Only one child made the pattern GRRRGRGRRRGR stating that for this instance the repeat was GRRRGR. The majority of this class could represent the pattern (a) in a variety of different ways, for example □□◇□□◇

Phase 2: In this phase children were again asked to expose subsequent repeats for the GGRGGR pattern, (e.g., GGR; GGRGGR; GGRGGRGGR etc.), count the number of greens and reds in the exposed patterns but this time not only record the number of greens and reds in a table but also the number of exposed repeats. Figure 3 illustrates a typical response of how children recorded the data and described the pattern in the table.

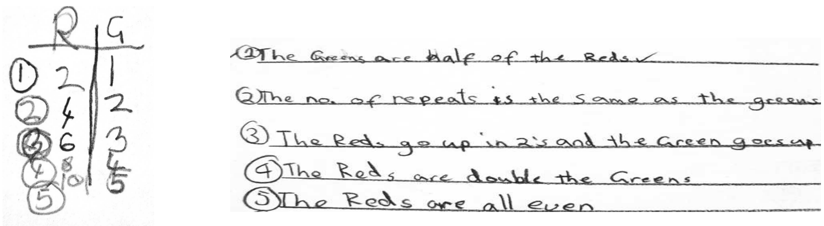


Figure 3. An example of a response given for the GGRGGRGGR pattern.

In this phase, a discussion ensued about the types of patterns that were in the table, across patterns and down patterns. As Brian commented, *Why do we need the number of repeats? It is the same as the last column.* Children were directed to write the words ‘across’ and ‘down’ on their papers and to find two for each. This seemed to assist them in focusing on a wider variety of patterns. Also the students’ responses were classified in the subsequent classroom discussions with regard to the patterns they could see in the table as either ‘down patterns’ or ‘across patterns’. Students were asked to justify their responses.

Phase 3 and Phase 4 were similar to the phases that occurred in Classroom 1, only in this instance children were more capable of finding uncountable steps from the patterns they described in the table. It was conjectured that this was assisted by recording the repeat number as well as the number of reds and number of greens in the table, and by the discussions that directed the children to identify down and across patterns. Many appeared to use the across pattern to assist them to find the solutions to uncountable steps.

Lesson 2

The focus in this lesson was to (a) implement strategies that were successful in Sarah’s class and determine if they assisted Amy’s class in reaching generalisations with regard to the patterns in the table, (b) introduce ratio and rate, and (c) ascertain if children in Sarah’s class could describe generalisations and record their generalisations using abstract symbol systems.

Classroom 1 – Amy’s class

The lesson began with a refocus on the pattern RRRGRRRGRRRGRRRG. The introduction of the separate column for recording the number of repeats and the insistence that they look for ‘across’ as well as ‘down’ patterns appeared to assist these children in describing the generalisations as well as reaching correct solutions for uncountable steps. From an examination of their responses on their work sheets, 15 children attempted to write across patterns, with 4 children linking the repeat number to the number of greens and 11 children linking the repeat number and the number of greens to the number of reds. This result suggested that recording the repeat number and separating the generalisations into across and down patterns assisted these children to find across patterns. Twenty one children also successfully ascertained that for 100 repeats there are 300 reds and 100 greens. Ten children successfully recorded that if there are 60 reds, there are 20 greens and 20 repeats. A further 13 simply stated that for 60 reds there are 20 greens. Twenty two children could also record the relationship between the number of reds and greens as ratios (e.g., 3 reds to 1 green, 6 reds to 2 greens etc.).

Classroom 2 – Sarah’s class

The instruction in this class extended to the ratio phase and included an examination of the repeating pattern RRGGGRRGGGRRGGGRRGG and writing this ratio in general terms. The question was asked, “Suppose I had n repeats, what is the ratio of reds to greens. How many reds? How many greens?” They were asked to write their responses on the back of their worksheet. The following Figure illustrates some of the children’s responses to this question.

When asked to explain their response (d), Annabelle said, *You add another leg to n for reds and then another leg for greens.* Thirteen children wrote responses similar to those represented in Figure 4, four wrote responses in terms of large numbers (e.g. 2000 reds to 3000 greens), and four did not attempt to write a generalisation.

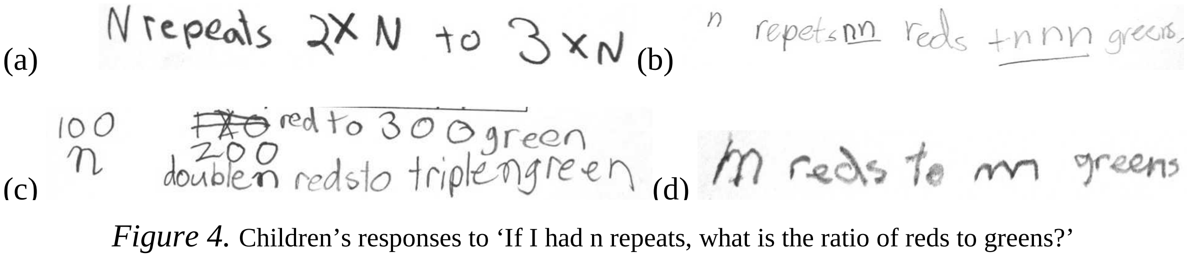


Figure 4. Children’s responses to ‘If I had n repeats, what is the ratio of reds to greens?’

Discussion and Conclusions

This research not only commences to document young children's thinking about repeating patterns, but also instructional processes that begin to assist in broadening their thinking about repeating patterns. It also includes reasoning of these patterns in general terms and relating them to concepts such as ration and proportion. Four conclusions are drawn from the data.

First, the results of the pre-test indicated that these children, after their experiences in the early years had a significantly greater understanding of repeating patterns than growing patterns, indicating that either growing patterns are cognitively more difficult, or their classroom experiences in the early years focussed predominantly on the exploration of repeating patterns. Following discussion with the classroom teachers and the children themselves, it seemed that in one of these schools the later was the case. This is a concern as it is the growing patterns that are traditionally used to bridge the gap between arithmetic and algebra in early adolescent classrooms. Hence, it is conjectured that many children may be experiencing difficulties with this transition due to their lack of prior knowledge and experiences with growing patterns as well as difficulties with co-variational thinking.

Second, many of these children viewed repeating patterns as having a particular starting point, implying that they do not see repeating patterns as extending in both directions. This is evidenced by their belief that RGGRGGRGG and GGRGGRGGRGGR are different patterns. Yet this understanding underpins our discussions with regard to patterning the number line to include the negative numbers and the place value chart including the decimals. Further research on the impact that this has on these conversations is required.

Third, past research has indicated that children tend to have a propensity to look for the additive strategy (look down the table) when searching for patterns in tables of values (Warren, 1996). This research confirms this finding. It also suggests that particular teaching strategies and questions can assist young children to begin to search for 'across' patterns. This is evidenced by the change in conversations between Lessons 1 and 2 in Amy's class. After establishing structures that assisted these children to refocus their pattern searching activity, most could give at least one across pattern. This refocus also assisted these children to correctly answer questions with regard to uncountable steps in the pattern, for example, how many reds and greens in 100 repeats? The instruction also appeared to assist children such as Kyla to reach beyond generalising from one example to uncountable steps.

Fourth, there has been an assumption that young children cannot express generalisations with more abstract symbol systems. This research suggests that they can. Their generalisation seemed to fall into four main categories, namely (a) using large numbers to express the generalisation, (b) simply repeating the number of n's to form an n, two n's joined together and finally 3 n's joined together, , (d) using words to express the generalisations, such as, Double n and triple n or two times n and three times n, and (e) formal notation, such as, 2xn 3xn. None of these children placed the number after the variable (e.g. n2 or nx2), a common problem delineated in past research. It is interesting that after reanalysis of the transcripts it appeared that the classroom conversations always placed the number before the variable (e.g. double the number of n's or 2 times the number of n's). The role that language plays in assisting children record their generalisation in the correct mathematical order deserves further investigation.

This research is ongoing. The results from these particular lessons not only gives future directions for the research but also commences to identify teacher actions that assist young children to generalise and formalise their mathematical thinking, and identify thinking that impacts on this process. Many of the difficulties these children experienced mirror the difficulties found in past research with young adolescents. This suggests that perhaps these difficulties are not so much developmental but experiential. From this beginning research it seems that young children can begin to articulate pattern structure sin general terms.

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