Primary Students' Knowledge of and Errors on Number Lines

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This paper reports on fifth graders' proficiency with number line tasks in an interview situation. The results revealed that at least 10% of students were unsuccessful in using a simple number line effectively. Additionally, some students' explanations suggest that they do not appreciate that the number line is a measurement model rather than a counting model. This study concludes with recommendations for explicit instruction, a note of caution for interpreting number line items on numeracy tests, and avenues for future research.

Introduction

The number line is a commonly used instructional aid in the primary years and is often featured in mathematical tests. Hence, the number line is assumed to be a tool that supports the development of conceptual understanding and an adequate measure of mathematical understanding. However, in calling for research on mathematical practices to improve equity for students and improve performance, Ball (2004) argued that there is a need for "attention to aspects of mathematical proficiency that are often left implicit in instruction, going beyond specific knowledge and skills to include the habits, *tools*, dispositions, and *routines* that support competent mathematical activity" (p. 11) (emphasis added). Hence, this paper explores students' proficiency with the number line in mathematical activity.

The Number Line

The number line is a diagram in which single positions encode quantitative information by their position on a horizontal or vertical axis (Mackinlay, 1999). The unidimensionality and encoding technique of number lines distinguishes them as an "Axis language" from the five other key graphic languages, namely Opposed-position languages (e.g., bar chart), Connection languages (e.g., network), Map languages (e.g., topographic map), Retinal-list languages (e.g., graphics featuring colour, shape, size, saturation, texture, orientation), and Miscellaneous languages in which information is encoded with a variety of additional graphical techniques (Mackinlay, 1999; see also Lowrie & Diezmann, 2005).

Number lines have three potential cognitive advantages for users. Firstly, they accommodate *mathematical variability* of concepts. Dienes (1964) argued that many mathematical concepts are essentially multi-dimensional and particular representations illustrate specific aspects of a concept. For example, a number line is useful in showing the continuity aspect of rational numbers. Secondly, number lines are part of a suite of visual representations that contribute to the *perceptual variability* of a concept. According to Dienes (1964) it is advantageous to have different representations of the same concept. For example, fractions can be represented by a number line and a pie diagram. Thirdly, number lines are *a tool for representational transfer*. Representational transfer occurs when tasks make use of a common representation, and the solution procedure is derived from the representation (Novick, 1990). Thus, in representational transfer "the primary goal is transfer of a representation in the *absence* of a common solution procedure" (Novick, 1990,

p. 130). For example, by considering how to use a number line to find a missing number in a set of whole numbers, a number line might cue a student about how to find a missing number in a decimal sequence.

Number lines with marked line segments are referred to as structured number lines. Advocates of these number lines argue for their value in number sequencing activities (e.g., Wiegel, 1998). However, number sequencing on a number line goes beyond knowing the order of number names. Because the number line is a measurement model, rather than a counting model, numbers on the number line are representations of lengths rather than simply the points they label (Fuson, 1984). Thus, in determining an unknown marked position on a number line, the proximity of the unknown from the known numbers is important. For example, on a number line that commences with 0 and concludes with 10, the number corresponding to a marked position between these numbers depends on its proximity to each of the numbers. Additionally, number lines reportedly have the capacity to concretise mathematical operations. For example, Davis and Simmt (2003) reported that one student interpreted 3 x -4 as "three hops of length four" along a number line and concluded that this student's "concept of multiplication as repeated addition was blended with the concept of multiplication as movement along a number line (p. 158). However, the efficacy of the structured number line cannot be assumed because contrary results have also been reported. For example, Fuson, Smith, and Cicero (1997) conducted a year-long teaching experiment with a class of first graders in which they trialled various conceptual supports for learning single digit addition and subtraction. They concluded that the number line was neither "particularly powerful nor interesting to the children (p. 748)." Thus, there is a need to better understand the conditions under which a structured number line is effective.

Structured number lines often feature on numeracy tests however their utility as a measure of rational number knowledge needs to be thoughtfully considered. Ni (2000) questioned the validity of number lines as a measure of rational number knowledge on the basis of her study with 413 fifth and sixth graders in which she found that number line test items were poor indicators of children's understanding of fractions. She argued that we should not automatically attribute students' poor performance to a lack of knowledge of what is being measured and proposed that an alternative plausible solution for poor performance could be that the measurement process does not tap students' knowledge of what is to be measured. Ni proposed that the utility of an item in tapping particular knowledge provides an explanation for the apparent discrepancy between children's knowledge of basic properties of rational number as reported in developmental studies compared to educational assessments. Thus, the use of a number line in assessment as an effective measure of mathematical competence on a particular topic warrants further exploration.

There are also empty number lines (Gravemeijer 1994). An empty number line is a relatively new didactic model, which is reportedly "a very powerful model for the learning of addition and subtraction up to 100" (Klein, Beishuizen, & Treffers, 1998, p. 443). The success of the empty number line is due to its modelling function and the interactivity generated from student-constructed number lines (Klein et al., 1998). Its failures as a model has been attributed to students' lack of flexibility with the model and its association with measurement (Gravemeijer, 1994). Thus, the lack of foundation in measurement and the lack of convention make the empty number line a fundamentally different model from the structured number line, which is the focus of this study.

Design and Methods

This investigation is part of a 4-year longitudinal study in which we are monitoring the development of primary students' ability to decode the six types of graphical languages including Axis languages (e.g., the number line). Elsewhere, we have documented primary students' knowledge of particular graphical languages and their relative difficulty (Lowrie & Diezmann, 2005). The aims of this study were:

1. To ascertain the proportion of students who were successful on two structured number line tasks;
2. To identify the knowledge that led to successful use of the number line; and
3. To document the errors that led to unsuccessful use of the number line.

The Participants

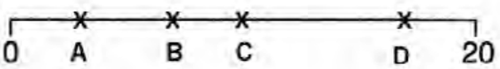
The participants were 67 Grade 5 students (aged 10-11 years) from class groups at Overton (n = 24) and Stanley (n = 43), which are two primary schools in a moderate socio-economic area of a capital city. Overton is a public school (N = 393), and Stanley is a parochial school (N = 685).

The Interview

The interview tasks were the easiest pair of Axis language items (See Figure 1) drawn from the 36-item Graphical Languages in Mathematics [GLIM] test (Lowrie & Diezmann, 2005), which comprises six sets of graphic items for each of six graphic languages. The two selected Axis items are similar in that they focussed on the identification of unnumbered positions on a number line and dissimilar in that Item 1 and 2 focussed on whole numbers and decimals respectively. The students completed these two items during an individual interview and then explained their thinking.

1. Estimate where you think 17 should go on this number line.

(QSCC, 2000a, p. 11)



2. Colour a bubble to estimate where you think 1.3 should go on this number line.

(QSCC, 2000b, p. 8)

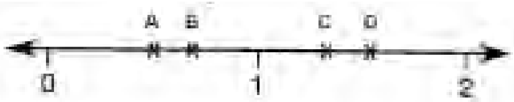


Figure 1. Axis items.

Data Collection and Analysis

The interview data compromised students' multiple choice selections and the reasons students gave for their answers. The interviews were video- and audio-taped to facilitate analysis. Aim 1 will be achieved by analysing the frequency of students' performance at each school on each task. Aims 2 and 3 will be addressed by analysing the reasons for successful and unsuccessful responses respectively and their relative frequencies.

Results and Discussion

The first aim of the investigation was to determine the proportion of students who were successful on each item. The results show that on average between 80% and 91% of students were successful across tasks and schools with means higher for Item 1 than Item 2. The former focuses on whole number and the latter on decimals. There were two points of interest. Firstly, the extent of the gender difference was unexpected. With the exception of Item 2 for Overton, there was between 9% and 13% performance difference in favour of males (See Table 1). This result is consistent with the results of another cohort (N = 217) from the larger study across the six Axis items including the two items discussed here (Lowrie & Diezmann, 2005). In that analysis, the Axis items were the only set of graphical languages that revealed a statistically significant difference between the mean scores of male and female students across the total set of any of the graphic languages [*Axis* ($t = 12.2$, $p \leq .001$)]. Secondly, there was a substantial performance difference across schools with Stanley (n = 43) outperforming Overton (n = 24) on both items. A plausible explanation for the higher performance of one school over another is the *perceptual variability* of the curriculum (e.g., Moss & Case, 1999).

Table 1.
Overview of Successful Student Performance on Axis Items

	Item 1			*Item 2*		
	Male	Female	**Total**	Male	Female	**Total**
Overton	90%	78.6%	**83.3%**	80%	78.6%	**79.2%**
Stanley	95.2%	86.3%	**90.6%**	95.2%	81.8%	**88.3%**

The second aim of the investigation was to identify the knowledge that led to successful use of the number line. Successful students gave seven reasons for their selection of responses for either Item 1 or 2 (See Table 2). These reasons can be grouped into two categories. The *Measurement* category consists of those reasons that indicate an understanding of the number line as a measurement model through explanations that refer to distance, proximity or reference points (i.e., CI, EP, LR, RS, RP). The *Inappropriate* category comprises explanations that focus solely on counting (CO) or guessing (GU). On Item 1 (whole numbers), all successful students from Overton and Stanley gave reasons from the Measurement category. However, on Item 2, there were substantial differences between Overton and Stanley. All successful students from Overton gave Measurement reasons. However, at Stanley, some students gave Measurement reasons (81.5%) and others gave Inappropriate reasons (18.5%). One interpretation of these results is that as the difficulty of the number line item increases from whole numbers to decimals some students were less able to provide an appropriate reason for their selection of response.

Table 2
Frequency of Successful Performance on Items 1 and 2

Reason	**Example**	**Overton** *Item 1* (n = 20)	*Item 2* (n = 19)	**Stanley** *Item 1* (n= 39)	*Item 2* (n = 38)
MEASUREMENT					
closest to an item (e.g., number) [CI]	*I chose D because it's closest to 20 and C is too far away.	75.0%	47.4%	66.7%	28.9%
estimating position [EP]	*I chose D because B is right, a bit far away from 20 and C is in the middle and I thought that would be about 10 and A would be too close to the 0 to be 17.	15.0%	10.6%	10.3%	21.1%
using a letter or number as a reference point [LR]	*I think it would be D because if half of that number line is number C and you would imagine the number line in the middle and then you can just look further up to 20 …	5.0%	21.1%	10.3%	21.1%
relative amount of space [RS]	*Because of the amount of space between each letter and the amount of space between D and 20.	5.0%	5.3%	2.6%	00.0%
to right or past or after 1 [RP]	* I chose C which is a bit to the right of the 1 and I thought it would be there 'cause it's closer to 1 than D is cause I think D would be 1.5.	0%	15.8%	0%	10.6%
INAPPROPRIATE					
counting on or back [CO]	*I think it should go there (D) because it's next to 20 and it goes 19, 18 then 17.	00.0%	00.0%	10.3%	10.6%
Guessing [GU]	#Well, it was a toss up between C & D and I chose C.	0%	0%	0%	7.9%

Key: *= Item 1 reason; # = Item 2 reason

Table 3
Frequency of Unsuccessful Performance on Items 1 and 2

Error	Example	Overton		Stanley	
		Item 1 (*n* = 4)	*Item 2* (*n* = 5)	*Item 1* (*n* = 4)	*Item 2* (*n* = 5)
SOLUTION					
focusing on distance (between, too far, too close) [FD]	*I chose C because that would be too close.	0%	0%	25%	0%
inappropriate counting back with each letter as a number [IC]	*I think it should be C because I reckon 19 would be about there. That would be 18 on D.	100%	0%	50%	20%
inaccurate position [IP]	#I did D because it's sort of closer to 1 than 2 and it's sort of in the middle as well to put 1.3.	0%	60.0%	0%	0%
misreading the diagram [MD]	#I said A because it's kind of half way in between the zero and the 1 and the B is a bit more like 4 so I just said A cause it's about half way.	0%	40.0%	0%	40%
EXPLANATION					
guessing [GU]	*I just guessed because I didn't really get it.	0%	0%	25.0%	0%
vague answer [VA]	#…because that would be one less and I thought that would be that too so I thought that would be good.	0%	0%	0%	40%

Key: *= Item 1 reason; # = Item 2 reason

The third aim of the investigation was to establish the errors that unsuccessful students made in their use of the number line because knowledge of errors is an important facet of pedagogical content knowledge (Carpenter, Fennema, & Franke, 1996). The range of errors identified is shown on Table 3 in two categories. *Solution* errors comprised difficulties with distance (FD), position (IP), counting (IC) or misreading the diagram (MD). The predominant Solution error across items and schools was inappropriate counting (IC). This error supports Fuson's (1984) concern that measurement foundation of the number line is overlooked by students and teachers. *Explanation* errors consist of guessing (GU) and vague answers (VA). Explanation is a fundamental mathematical practice and students need to become adept at explaining their solutions (e.g., Diezmann, 2004). Hence, to guess or give a vague answer is a general error because it indicates a lack of understanding of acceptable mathematical practice. Students need to be encouraged to provide adequate explanations because the communication of an explanation provides them with an opportunity to review and, if necessary, refine their mathematical thinking. For example, Shaun (Stanley) realised his error on Item 2 during his explanation for selecting D: "Because it's the closest to number one and D might be 5 [meaning 1.5]. Oh because um the one is there and I *accidentally* put it down near this area and it should be one point three [1.3]" (emphasis added).

Conclusion and Implications

Visual representations are an important tool for thinking and communicating mathematically (e.g., Goldin, 1998) and essential for the Information Age (Cazden et al., 1996). Hence, students need to become proficient with a broad repertoire of visual representations in one, two and three dimensions. The investigation of students' proficiency with the number line, a one-dimensional representation, provided an opportunity to determine the evidence to support intuitive use of the number line in instruction and to validate its role in assessment.

This study informs instruction in three ways. Firstly, it is fallacious to assume that students are proficient users of number lines even for the seemingly simple task of identifying unnumbered marks. At least 10% of students were unsuccessful on each of the number line items. Thus, students need explicit teaching about the number line. Secondly, the successful and unsuccessful students' responses indicate the importance of students' appreciation that the number line is a *measurement* rather than a *counting* model. Thus, instruction needs to emphasise the linearity rather than cardinality of the model. Thirdly, the role of explanation as a metacognitive process needs to be highlighted.

The study prompts us to provide a note of caution in relation to the interpretation of performance on number line items on state, national or international numeracy tests. This study and others (e.g., Ni, 2000) have identified that success on number line items involves both mathematical content knowledge and representational knowledge of the structured number line. Hence, to make claims about students' knowledge of a particular concept (e.g., decimals); there is a need to assess the concept using various representations (i.e., perceptual variability). Similarly, to make claims about students' knowledge of the number line, there is a need to assess the representation using various concepts (i.e., mathematical variability). Thus, the purposes of number line items on a test need to be clear.

Three avenues for further investigation have emerged from the study. Firstly, performance differences between groups warrant investigation. Reasons for the gender difference in favour of males and the school difference in favour of Stanley need to be understood. Secondly, the role of the structured number line for purposes apart from the identification of unnumbered marks needs to be explored. For example, number lines are used in operations and problem solving. Thirdly, the relationship between knowledge of the structured and empty number lines needs to be examined. The structured number line is widely used and the empty number line purportedly offers much promise. However, they appear to be fundamentally different models. Hence, the compatibility of these two models of representation needs to be established.

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