The Role of Abstraction in Learning about Rates of Change

Ibrahim Hassan
Macquarie University
chubby2020@lycos.com

Michael Mitchelmore
Macquarie University
mike.mitchelmore@mq.edu.au

Fourteen Year 11 advanced mathematics students participated in individual teaching interviews designed to investigate how they learnt various rate of change concepts. The theoretical framework compared two models of abstraction: the empirical abstraction model of Mitchelmore and White and the nested RBC model of Hershkowitz, Schwarz, and Dreyfus. Examples of learning were found that fitted the nested RBC model, but none that fitted the empirical abstraction model. It was concluded that the nested RBC model is valuable for understanding student learning of the concepts of average and instantaneous rate of change, but that empirical abstraction is likely to be more valuable in understanding how students develop a global concept of rate of change earlier.

In the past six years or so, two models of abstraction in mathematics have been advanced. One, the *empirical abstraction* model, assumes that students make abstractions as the result of recognising underlying similarities between superficially different contexts that are familiar to the student (Mitchelmore & White, 2004). The other, the so-called *nested RBC model*, assumes that students construct new abstractions by reorganising existing abstract concepts (Hershkowitz, Schwarz, & Dreyfus, 2001). To date, there have been no studies comparing the two models.

Rate of change is a mathematical concept which lends itself to the comparison of the two models. It occurs in many everyday contexts, but its mathematical treatment —leading to calculus—can be very abstract and difficult for students (White & Mitchelmore, 1996). Moreover, there have been few studies of how students learn the concept. The study reported in this paper investigated the question of whether either of the two models of abstraction can help us understand how students learn the concept of rate of change.

Abstraction

Abstraction has many facets and there is no consensus among researchers with regard to a unique meaning (Hazzan, 1999). But all agree on two aspects: (1) A new mental object is created as a result of an abstraction process. For example, Mason claims that abstraction in mathematics is a common experience, "an extremely brief moment which happens in the twinkling of an eye; a delicate shift of attention from seeing an expression as an expression of generality, to seeing the expression as an object or property" (1989, p. 2). (2) This new object separates out some relevant features from others considered irrelevant. Thus Davidov (1990, p. 13) defines abstraction as the process of "separating a quality common to a number of objects/situations from other qualities".

Where researchers differ is on the abstraction process itself.

The empirical abstraction model. Skemp's (1986) defines abstraction as "an activity by which we become aware of similarities among our experiences" (p. 21). Mitchelmore & White (2004), following Skemp, claim that the first phase of the abstraction process is the recognition of common features in a variety of different situations. In everyday experience these features may be superficial (e.g., colour), but in mathematics they are always structural (e.g., number). In the second phase, the similarity that has been recognised becomes abstracted and forms a concept which in a sense embodies that similarity. The empirical abstraction model has been applied to several school topics, including calculus (White & Mitchelmore, 1996).

The nested RBC model. Hershkowitz et al. (2001) define abstraction as "an activity of vertically reorganising previously constructed mathematics into a new mathematical structure" (p. 202). The term "vertical", following the Dutch school (Treffers, 1987), is intended to indicate that the new concept exists on a higher level than the concepts from which it is constructed. Their model of the abstraction process comprises three *epistemic* (knowledge-building) *actions*: recognition, building-with and construction (hence "RBC"):

- *Construction* was identified as the "central step of abstraction". The authors argued that "novelty implies construction" and "when a novel structure 'enters the mind', it has to be cognised, or pieced together from components, usually simpler structures" (p. 212).
- *Recognition* relates to structures that one had "presumably used previously in other situations and was able to adapt, at a structural level, to the present situation and make use of them as needed" (p. 213). However, "whereas for the experts, the process is a matter of recognising, for a suitable, prepared novice, it might be an opportunity for engaging in a process of constructing a deep structure" (p. 214).
- *Building-with* was defined as "combining structural elements to achieve a given goal" (p. 215). It is this combination of recognised concepts and relationships that may eventually lead to the construction of a new abstract concept.

Hershkowitz et al. (2001) also state that "the action of constructing does not merely follow recognition and building-with in a linear fashion but simultaneously requires recognition of and building-with already constructed structures"(p. 218) and refer to the "dynamic nesting of the epistemic actions" (p. 218).

The nested RBC model has also been applied to several school topics, including the sketching of function graphs (Ozmantar & Roper, 2004).

Rate of Change

Rate of change may be defined as how one quantity changes in relation to another (Barnes, 1991). The importance of rates of change in our lives is well documented (Hauger, 1995), and can be illustrated by a diverse range of examples including speed, percentage increases, growth and decay of populations. It is also one of the core concepts of calculus, where tabular and graphical representations give way to symbolic methods.

Hauger (1995, p. 10) identified three essential aspects of the rate of change concept as global (macro qualitative), interval (macro quantitative), and point-wise (micro qualitative):

- The *global* aspect focuses on how the dependent variable changes with respect to the independent variable without using numerical values to describe those changes. Understanding would include the recognition of zero, constant, positive, negative, large, small, increasing and decreasing rates of change
- The *interval* aspect comprises the concept of an *average rate of change* (AROC) over a period. The AROC over an interval is defined as the change in the dependent variable relative to the change in the independent variable, represented graphically by the gradient of the chord joining the two end-points.
- The *point-wise* aspect refers to the concept of *instantaneous rate of change* (IROC) at a particular value of the independent variable. The IROC may be defined loosely as the AROC over a "very small" interval around the given point, represented

graphically by the gradient of the tangent at that point. The interval has to be small enough that the rate of change, viewed globally, is more or less constant (within a certain degree of error).

In all aspects, data may be presented in tabular, graphical or algebraic form. The current study is restricted to data presented as tables or graphs.

Abstraction in learning rate of change. The process of learning about rate of change clearly involves both empirical and theoretical abstraction. Students need to see the same idea in different realistic contexts if it is to be meaningful to them. On the other hand, the definition of IROC is unrealistic in the sense that "sufficiently small" intervals can often not be constructed in practice (for example, when the independent variable is a whole number). The two models of abstraction may therefore be of different applicability.

According to the empirical abstraction model, students would construct the various aspects of the rate of change concept by recognising similarities between different, familiar rate of change contexts and then abstracting the similarities. For example, recognising that cars increase and decrease speed, that the price of petrol goes up and down and that the value of a car depreciates over time would all contribute to a global understanding of rate of change. Students may also meet the ideas of average speed for a journey, average rate of increase in petrol price over a year, or average depreciation rates over 5 years, all of which are easily calculated from numerical data, and hence form a general concept of AROC by empirical abstraction. Everyday examples of IROC would be rarer, the most obvious example being instantaneous speed as shown by a speedometer. However, the instantaneous rates in such examples come from a "black box", and so would not help students understand the relation between IROC and AROC.

In the nested RBC model, students would learn the various aspects rate of change by applying, integrating and reorganising several more elementary, known concepts. For the global aspect, these elementary concepts could include such ideas as constancy, size and change. For AROC, they could be difference and ratio. For IROC, they could be AROC, linearity and tangency. However, for none of these three aspects is it clear how precisely students would reorganise the various subconcepts into a new concept.

This Study

This study investigated how Year 11 students learnt about rate of change by observing them as they worked individually through a series of problems embedded in a variety of familiar contexts. The procedure was designed to allow the identification of the abstraction process, if any took place, either according to the empirical or the nested RBC model. We sought answers to the following questions:

- How well does each of the two models of abstraction describe the process of learning about rates of change?
- What can we infer about the learning of rate of change concepts?

Methodology

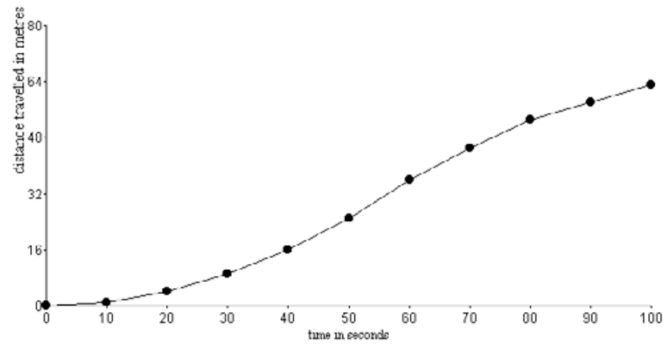
Participants. The sample comprised 14 volunteer Year 11 students, 8 males and 6 female, from five high schools in Sydney, New South Wales. All of them had been exposed to an initial introduction to calculus prior to participating in this study.

Procedure. Each student in the sample was interviewed twice by the first author. The first interview lasted an hour and the second, which was conducted a week later, half an hour. Both interviews were audio recorded and transcribed.

Interview items. There were four items used in the first interview (Items A-D). Only one item was used as a follow-up in the second interview (Item E). Except for Item D, all items were designed to assess students' understanding of the three aspects of rate of change discussed above. In Items A-C, where students showed that they did not understand any aspect, the interviewer attempted to teach them the appropriate concept.

Items A-C and Item E comprised representations of real-world relationships and all had the same basic structure. As an example, Item A is presented in Figure 1.

The graph below shows the distance travelled by a vehicle over time. The distance is measured in metres and is denoted by the *y*-axis and the time is measured in seconds and is given by the *x*-axis.



(a) What can you say about the speed of the vehicle?
(b) When was the vehicle driven the fastest?
(c) What is the average rate of change in the distance travelled from the 40th to the 50th second?
(d) What can you say about the rate of change in the distance travelled at the end of the 40th second?

Figure 2. Interview Item A.

Item A was designed to test students' specific understanding of the concept of speed and their ability to relate that understanding to the concept of rate of change. It was anticipated that the context, and the graphical representation, would be familiar to most students. Parts (a) and (b) related to global aspects of rate of change, part (c) to AROC, and part (d) to IROC.

Item B was based on the concept of population growth. Students were presented with two tables of values: one reflected a constant decline in a crocodile population whereas the other showed a situation where a decline gradually slowed and was replaced by an increase. Students' understanding of rate of change was tested in both situations separately before asking them to compare the two different situations.

Item C was about a cooling experiment. In part (c), the end-points for the AROC had to be interpolated from a table of values. Item E, the only item in the second interview, was about the growth in a student's height over several years and did not require interpolation.

The structure of Item D was different. This item comprised several questions intended to summarise what students had learned from working through Items A, B and C. The first question asked if they had recognised anything similar in the three items. The remaining questions asked them how to find an AROC and an IROC, and to explain the difference between the two.

Teaching. For students who did not seem to understand AROC, it was explained that the AROC was the ratio (presented as a fraction) of the total change in the dependent variable to the total change in the independent variable over a fixed period. The significance of the end points of the interval was emphasised.

IROC was taught using a zooming-in approach. To avoid having to teach students how to use a graphical calculator or computer software, students were shown a series of graphs of the given relationship graph where the domain and range of each variable were successively reduced. Students were asked to examine the behaviour of the graph around the given point and led to see that the initial non-linear graph appeared to become gradually linear and the rate of change therefore approximately constant. The student was then asked to imagine this straight line extended over a greater domain and to predict where this line would be after zooming out back to the original graph. It was expected that students would recognise the line as the tangent to the graph at the given point.

Analysis. Students' responses to the various test items were categorised firstly in terms of students' actions and secondly as to whether these actions demonstrated prior understanding (not followed by teaching) or subsequent learning (after teaching) for each of the three aspects of the rate of change concept. Strict criteria were established to determine whether a student had successfully learnt a concept either with or without teaching; see Hassan (under examination) for details. A search of the transcripts was then made in any attempt to identify clear examples of abstraction according to either the empirical or the nested RBC model.

Results

Overall. Table 1 shows the number of students who showed that they understood the three aspects of rate of change embedded in each item, without the need for any teaching.

Table 1
Number of Students who Successfully Learnt Concepts without Teaching (N=14)

Item	Global	AROC	IROC
A	13	3	3
B	14	8	10
C	14	7	14
D	-[a]	14	14
E	14	10	11

[a]There were no questions referring to global rate of change in Item D.

Only one student showed no global understanding of rate of change, and then only on the first item. This student seemed fixed on applying the "distance ÷ time" formula to find speed and was unable to identify the maximum speed from the graph. However, she had no difficulties with later items.

By contrast, few students showed an understanding of either AROC or IROC at the start of the interviews. Surprisingly, it took students longer to learn about AROC than IROC, but by the end of the first interview (Item D), all students correctly explained how to find these two concepts in both tabular and graphical representations and seemed to comprehend the difference between them. All recognised some similarities between Items A-C, but responses were often in terms of superficial characteristics such as graphical representation. Item E showed that several students had not fully understood AROC and IROC, but the overall performance level was far greater than for the initial item.

Identifying abstraction according to the two models. There was no examples in the transcripts of abstraction occurring in "the twinkling of an eye", as Mason (1989, p. 2) claimed. It was decided that evidence for abstraction could nevertheless be sought by examining the transcripts of students who learnt an aspect of rate of change in one item and used it successfully in all later items. Furthermore:

- If a student first learnt the concept in Item A, then abstraction could have occurred as described in the nested RBC model but would be unlikely to have occurred as described by the empirical abstraction model.
- Evidence for empirical abstraction should be sought among students who first learnt an aspect of rate of change in Items B or C.

A search of the overall results identified four students who could, accordingly to the above specifications, have learnt AROC by the RBC model and one who could have learnt it by the empirical abstraction model. There were six students who could have learnt IROC by the RBC model but none who could have learnt it by the empirical abstraction model. These transcripts were then examined in more detail. Several students were found to show clear evidence of learning according to the RBC model, but only one rather doubtful case of empirical abstraction was found. The following examples were the clearest (all student names are pseudonyms).

In Item A(c), after ascertaining that Lou had no general concept of AROC, the interviewer (I) showed Lou (L) that the AROC in the distance travelled from the 40th to the 50th second is found from the distance covered in that time.

I: Okay, let me explain. What's the change? Let's get some approximation values here. What's the distance covered during that time interval? Can you give me some rough idea?
L: Rough idea? Okay! It's around 16, say 13, 14, the gap between there, 13.
I: Okay, so that distance travelled is 13, and that is over…
L: [*interrupts*] 10 seconds. [*pause*] So to find the average isn't it 13 divided by 10?
I: Yes, it is.
L: I guess I just write it down. [*Lou writes 13/10 = 1.3 m/s.*]
I: That would give you the distance travelled …
L: [*interrupts*] in metres per second!
I: Yeah, that's good. So if we have an interval we can always work out the average rate of change over that interval.
L: So basically the average rate of change is asking me about the speed.
I: Yeah, average speed. […] So we find the difference in the distance travelled. And we find the difference in the time it takes and we divide one by the other to find the average rate of change. Is that understood now?
L: Yeah, Average rate of change is basically change between one to the other. So basically during that interval, find out the speed.

The transcript shows evidence that Lou *recognised* several concepts (including changes in the distance travelled, average, and speed both globally and in terms of the distance/time formula), she *built with* these concepts (i.e., made meaningful links among them), and she constructed a general AROC concept as the ratio "change between one to the other … during the interval". The transcript shows that she had no trouble with AROC in later items.

When IROC was taught to Dan in Item A(d), at first he was confused by the difference between "from the 40th to the 50th second" and "at the end of the 40th second". The interviewer went through the zooming process described above, and Dan was adamant that the graph will "appear to be a straight line, but is it really?" After a lengthy digression on how to find the rate of change for a linear graph, the interviewer asked Dan where the line would appear if you zoom out.

I: Well let's say I have extended this line so it is going all the way past 30.
D: Yes. So it would look like this [*sketches tangent with finger*]
I: Exactly. It would look like that. It would touch the curve at 40.
D: Yes, touch the curve at 40.
I: In fact there is a name for that. You call that line…?
D: Tangent.
I:. Exactly. […] So if you want to find the rate of change at a point, what we do is we draw a tangent to the graph at that point.
D: Oh, okay. So that tells us the exact rate of change, or speed, at 40 seconds.
I: Yes, and then we find the slope of that line. […]
D: Yes, you have to work out the slope of the tangent, and the slope of the tangent will give us the rate of change at 40 seconds.

There is clear evidence here that Dan had recognised a number of concepts, including linearity, tangents and slope. Then, with much help from the interviewer, he combined these concepts in a novel way and constructed an entirely new concept, the IROC.

The only case where something resembling empirical abstraction occurred was when Rav was taught AROC in Item B(c). When the interviewer pressed Rav to look for similarities to the previous item, at first she could not find any. But when pressed, Rav constructed an analogy between "change in distance divided by change in time" in Item A and "change in population divided by change in time" in Item B. She successfully followed the same analogy in later items and appeared to have constructed a general concept. However, although Rav's learning process involved searching for a similarity between the two contexts, it does not really fit the empirical abstraction model because her construction of an AROC in Item B was not meaningful *within that context*.

Conclusion

This study shows that the RBC model of abstraction has potential for helping educators understand the process of learning the quantitative aspects of rate of change—AROC and IROC. Firstly, the model underlines the importance of identifying the elementary concepts on which AROC and IROC are built. Secondly, it emphasises the need for students to have a sound understanding of those concepts before teaching the advanced concepts. Thirdly, it can indicate the path by which a teacher can lead a student who understands the elementary concepts to construct the more advanced concepts.

There was very little evidence in this study to support the use of the empirical abstraction model at this level. The students seemed to have already gained a fair global concept of rate of change, although they might not have learnt the term as such. If empirical abstraction takes place in the learning of rate of change, this clearly occurs before Year 11.

One inference from the results is that learning about IROC is not dependent on learning about AROC. Table 1 shows that IROC was slightly easier to learn than AROC, and there were a number of students who learnt AROC without having understood IROC. This was perhaps due to the way IROC was taught—by a method which avoided the traditional method of taking the limit of an AROC. The finding may nevertheless have interesting curriculum implications: It may be easier to progress from a global understanding of rates of change to IROC than to AROC.

In this study, some successful students seemed to have learnt the rote procedure "to find an IROC, you find the gradient of the tangent". Rav's superficial generalisation of the AROC was similar. As a result, in the follow-up interview (Item E), a few students used a tangent for AROC and a chord for IROC. However we approach abstraction, the danger of superficial generalisation must always be kept in mind.

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