Departing from the Traditional Long Division Algorithm: An Experimental Study

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The traditional long division assumes that users can apply a guess-and-match type mental process of searching for a maximum that is not greater than the dividend. This optimisation procedure frustrates some children because it confuses the concept of division that is embraced is not consistent with their life experiences associated with grouping and sharing. Initial results of our studies suggest that children taught by a new method perform better on a test than those who learn it through the traditional method.

Introduction

Teaching early grade students about the concept of division and the skill of performing a division algorithm is always a challenge to teachers. There are many studies which consider children's understanding of division in early elementary grades. The most common approach they learn for solving a computational division problem is the Long Division algorithm. It is well known that a student may perform correctly the long division and get the correct answer without knowing the essential meaning of division. Many teachers introduce division as a process of sharing (Liebeck 1990). The rules governing such a procedure of sharing, relate to groupings (Squire & Bryant 2003). During grouping, we must ask two fundamental questions: how many groups are involved in the sharing and how many can each group get? Traditionally, these approaches are termed "partition" and "measurement", respectively, although sometimes "quotation" is used instead of "measurement". Squire and Bryant (2002) investigated the influence of sharing on children's concepts of division by testing their ability to apply grouping within the divisor or within the quotient, considering the fundamental concepts of divisions by partition and measurement. We can summarise this in Table 1.

Table 1 *Division by Partition and Measurement*

Traditional Question	Name of Approach	Example
a ÷ b is interpreted to mean: "From the a objects, how many groups of b objects can we form?"	"Measurement" (sometimes referred to as "quotition")	12 divided by 3 equals 4 because the 12 can be divided into 4 lots with 3 objects in each.
a ÷ b is interpreted to mean: "If a objects are placed in b equal groups, how many will there be in each group"	"Partition"	12 divided by 3 equals 4 because if 12 objects are dealt to 3 people then each would get 4.

Zweng (1964) introduced the concept of rate to study how well children learned division in contexts that involved the measurement and partition interpretations of the operation (see Table 1). In performing a formal division such as $48 \div 6$, Robinson, Arbuthnott and Gibbons (2002) showed that adults tended to transform the division to multiplication by retrieving the product of 6 and 8 that is equal to 48. Ilg and Ames (1951) found that children might use related addition facts to solve division problems. Lakoff and Nunez (2000) noted that children conceptualized division in two ways: repeated subtraction and splitting objects into groups. Robinson and her colleagues (2005) reported that Grade 4 children used a repeated addition strategy and Grade 5 to 7 students performed division more by using a multiplication strategy. They referred to the division problem as a multiplication problem looked for the product of divisor and quotient to match the dividend. It was suspected that children might only infrequently access information they need from their times table because they had 'blind spots' in their times table knowledge.

Even though there have been several studies into how children deal with the meaning of division, procedurally and conceptually, many researchers find that teaching early-grade students about the concept of division is still a challenge to teachers, as they tend to perceive division as a process of sharing something that can be counted (e.g., Cowan & Biddle 1989; Miller 1984). From both computational practical perspectives, the concept of sharing can lead cognitively to two different cognitive understandings. Children think of: *"Take them all and so that there are none left over; at least take as many as you can"* and/or *"average them so that each gets an equal amount"*.

When attempting to use the traditional long division algorithm students have to think of a maximum digit that produces a *virtual product* (the product of the leading digit of quotient and the divisor, see the example of 216 ÷ 8 below) that is just less than the *virtual magnitude* represented by the corresponding digits of the dividend. Inevitably, many students find it difficult to relate the procedures to the meaning of sharing and averaging.

The division process involves how students handling the triangular relationship of three quantities: dividend, divisor and quotient. Let us go through possible mental processes involved in working on the following division

$$\begin{array}{r} 2 \\ 8\overline{)2\,1\,6} \\ \underline{1\,6} \\ 5 \end{array}$$

Guess the maximum possible 2-digit multiple of 8 that will give a result that is less than or equal to the leading 2-digit *number* "21". Then subtract 16 from 21 to get 5. Although 5 is less than 8, students have to bear in mind that this "5" is the tens-digit. 16 is the *virtual* product. Actually, $8 \times 20 = 160$ is involved in the first step.

$$\begin{array}{r} 2 \\ 8\overline{)2\,1\,6} \\ \underline{1\,6} \\ 5\,6 \end{array}$$

Carry down the last digit "6" to form the number 56. Now the digit "5" together with digit"6" has formed an actual remainder 56. That is larger than 8. The sufficient condition for further division (as an operation) is met.

$$\begin{array}{r} 27 \\ 8\overline{)216} \\ \underline{16} \\ 56 \\ \underline{56} \\ 0 \end{array}$$

Perform the final step by retrieving memory from the times table, that the product of 7 and 8 is 56.
It is obvious that the whole procedural division is unique, as “16” is the maximum possible multiple of 8 that less than “21” and 56 is uniquely the product of 8 and 7.

In most textbooks, the traditional long division approach is used. Students have to go through several difficult mental processes including:

1. Estimation of a suitable product(a virtual) via mental multiplication;
2. Retrieval of products from the times table to get a product (divisor×virtual quotient = virtual product, such that it satisfies the constraints that:
    a. the virtual product must be the maximum possible number (digits for the moment) by comparing its magnitude with the value represented by leading digits of the dividend;
    b. (b) if the trial product is the maximal one, one need to compare the remainder after subtraction of the leading digits of the dividend and the virtual product. Again, it is not a formal subtraction as we are doing subtraction of corresponding digits but not the actual numbers themselves.
3. Repeat the process again onto the remainder (digits only) if it is greater than the divisor

It is clear that step 2 is crucial and children must go through the difficult mental activity of optimisation. Finding such a maximum possible virtual product is almost always an obstacle to children. To them, the worst disadvantage is that the process is simply a set of rules for them to remember. They may have no understanding that the multiplication of divisor and the leading digits of the quotient give only a *virtual* product. The subtraction inside the long division working is not a complete subtraction because we neglect the role of the rest of the digits. This *partial* subtraction confuses students about the values of the digits and appears to be unrelated to the legitimate concept of grouping and sharing.

Many children rigidly remember to pull down the rest of digits from the dividend and perform the division continuously. Students do not know the actual meaning of why they are subtracting the leading digits only. In the example above the first remainder looks like a unit digit in the first stage but suddenly shifts to a tens-digit (as it should be) in the second stage of division (after carrying down the other digits of the dividend). Carrying down the final digit of the dividend is not fully justified mathematically because this digit itself is not a remainder as the division at the first stage did not involve it. Changing the role of the last digit to become the remainder can hardly convince students that it is a formal and fair sharing. That can lead to confusion about the concept about division. As a result, it may lead to difficulties in doing division of quantities of higher dimension, such as division of polynomials.

The above procedure is suggested by almost all textbooks. The rote memorisation of rules does not help those who have blind spots in the times table when they are attempting to calculate the maximum possible value of the virtual product at the initial stage of the division.

An Alternative Algorithm

We have developed an alternative method based on the previously stated idea of decomposing the dividend into a sum of smaller numbers such that we can perform the division on each of such elements of the partition. Borrowing the idea of the well known distributive law of *multiplication* over addition, we use a distributive law of *division* over addition. For example, $24 \div 4 = (12 + 8 + 4) \div 4 = 3 + 2 + 1$. Given that one might have the blind spot in the times tables of $6 \times 4 = 24$, one can decompose the 24 into groups of 12, 8 and 4 and divide each of 12, 8 and 4 by 4 to obtain 3, 2 and 1, respectively. The sum of the partial sums arising from the partitioning of 24 into 12, 8 and 4, and then dividing each of these by 4, is 6.

Instead of just a set of rules, we introduce this new method and call it the method of Partition of Quotient (POQ). Conceptually, we adopt the idea of grouping by divisor (Squire & Bryant 2002). It is basically a process of sharing. For example, sharing 12 objects among 4 people, each of the 4 people can get 2 objects first in the first round of sharing (grouping), and the remaining objects are shared again, so that each of them get one more. There is no need to estimate the maximum but only a number (partition of quotient) that produces a product for the intermediate step.

Even though the dividend has to be decomposed first in POQ, we must emphasise that this decomposition of dividend is not unique. In the procedure of division, the student is free to choose the partition that he or she can perform correctly when he or she knows any multiple of 4 that is less than 12. In the long division format, using the previous example of $216 \div 8$, we can write the procedures as follow:

$$
\begin{array}{r}
5 \\
\left.\begin{array}{r} 2 \\ 20 \end{array}\right\} \quad \boxed{20 + 2 + 5 = 27} \\
\hline
8\,\big)\,216 \\
160 \\
\hline
56 \\
16 \\
\hline
40 \\
40 \\
\hline
\end{array}
$$

Instead of applying a division rule on manipulating the digits, one may work on the division as an operation of finding a combination of multiplication such as $8 \times 2 = 16$, $8 \times 20 = 160$ and $8 \times 5 = 40$. The quotient will be $20 + 2 + 5 =$ 27. Basically, students may perform the division correctly even if they have a blind spot of $8 \times 7 = 56$. Note that 160, 16 and 40 are direct products. They are real, not *virtual*.

Alternately, one can write

$$
\begin{array}{r}
1 \\
\left.\begin{array}{r} 6 \\ 10 \\ 10 \end{array}\right\} \quad \boxed{10+10+6+1=27} \\
8\overline{)216} \\
\underline{80} \\
136 \\
\underline{80} \\
56 \\
\underline{48} \\
7 \\
\underline{7}
\end{array}
$$

Note that here {10, 10, 6, 1} is the partition and the leading two digits "13" in the remainder 136 is larger than the 8. This has already violated the rule of maximum match in the traditional method. The digit "6" is not a result of carrying down from the dividend 216. It is a direct result of subtraction. That is to say, there is no straight rule at all. The blind spot $8\times 7=56$ does not hinder the procedure to complete the division. It is intuitive to choose 10-10 in the first two steps because $8\times 10=80$ is relatively trivial. And the partition {10, 10, 6, 1} is of course not unique in terms of the number of elements as well as its order.

In teaching this method, a teacher should explain to the students that in sharing 216 objects among 8 people, each of them takes 20 objects first. That means that 160(160 is real product, not virtual) objects have been allocated. The rest of 56 objects have to be shared by the same 8 people again and if each takes 2, another 16 will be gone. Then there will be 40 left. And eventually, each person share 5 objects to complete the whole division process. Hence, each person has got, sequentially, 20, 2 and 5 objects. Therefore, the total number each person receives and shares will be the sum, that is $20+5+2=27$. Adding these three numbers here is no longer a straight rule to remember but a strong sense of the consequence of continuous sharing. This sense leads students to add the numbers to get the answer.{20, 2, 5} becomes a partition of the quotient 27 when dividing 216 by 8. In this process, we make use of the algorithm of POQ. When D is divided by d,

$D\div d=(q_1+q_2+q_3+\cdots+q_n)=Q,$ where $R_i\neq 0$ for $i=1,2,3,\ldots,n-1$ and $R_n=0,$ and

$D=q_1 d+R_1,\ R_1=q_2 d+R_2,\ \cdots,\ R_{n-2}=q_{n-1}d+R_{n-1},\ R_{n-1}=q_n d+0$

Substituting backward from the last equation by iteration will easily lead to the equation $D=(q_1+q_2+q_3+\cdots+q_n)\times d=Q\times d,$ where the R's are the remainders obtained in each step of POQ method in the lone division process. Here $\{q_1,q_2,q_3,\ldots,q_n\}$ is a partition of the quotient Q. Obviously it is not unique and it all depends on how familiar students are with the times table and how they choose the partitioned elements in the subsequent steps. For example, the division $24\div 4$, the partition can be either {4, 2}, {4, 1, 1} or {3, 2, 1}. Note that the student would not consider or intend to do the decomposition of the dividend 24 into 16 or 8. Instead, he or she may recall the multiplication of 4×4 =16, then divide the remainder 8 by 4 to give 2. Then adding up 4 and 2 gives the answer 6. We can see that a student can do the division even if he or she knows only the three products, namely $12=4\times 3,\ 8=4\times 2,\ 4=4\times 1$ but not the products $4\times 4=16$ and $4\times 6=24$. When applying the POQ method, we are not imposing a rule on students on the way to get the partition as it is not unique.

Methodology and Results

We have seven classes of Grade 3 students that have a total of 155 students. Their average age is around 8 years. We separated the classes into control (70) and experiment (85) groups. The same teacher taught the control group the traditional method and the experimental group the POQ method. We administered a pre-test comprising simple division problem to both groups. After a series of lessons involving around two hours of class instruction, a post-test was given to all students to test their performance on applying the learnt skills and skills they could transfer to a harder problems. Some students learning the POQ method could do the division on numbers of larger magnitudes, both the divisor and dividend. For example some could correctly work out $338 \div 26 = 13$ without prior knowledge of dividing a three-digit number by a two-digit one. The pre-test and post-test scores for the groups are shown in Table 2. In this data analysis, we did not take into account of the nesting of students within classes. That will be the next step of our investigation on the effectiveness of using POQ.

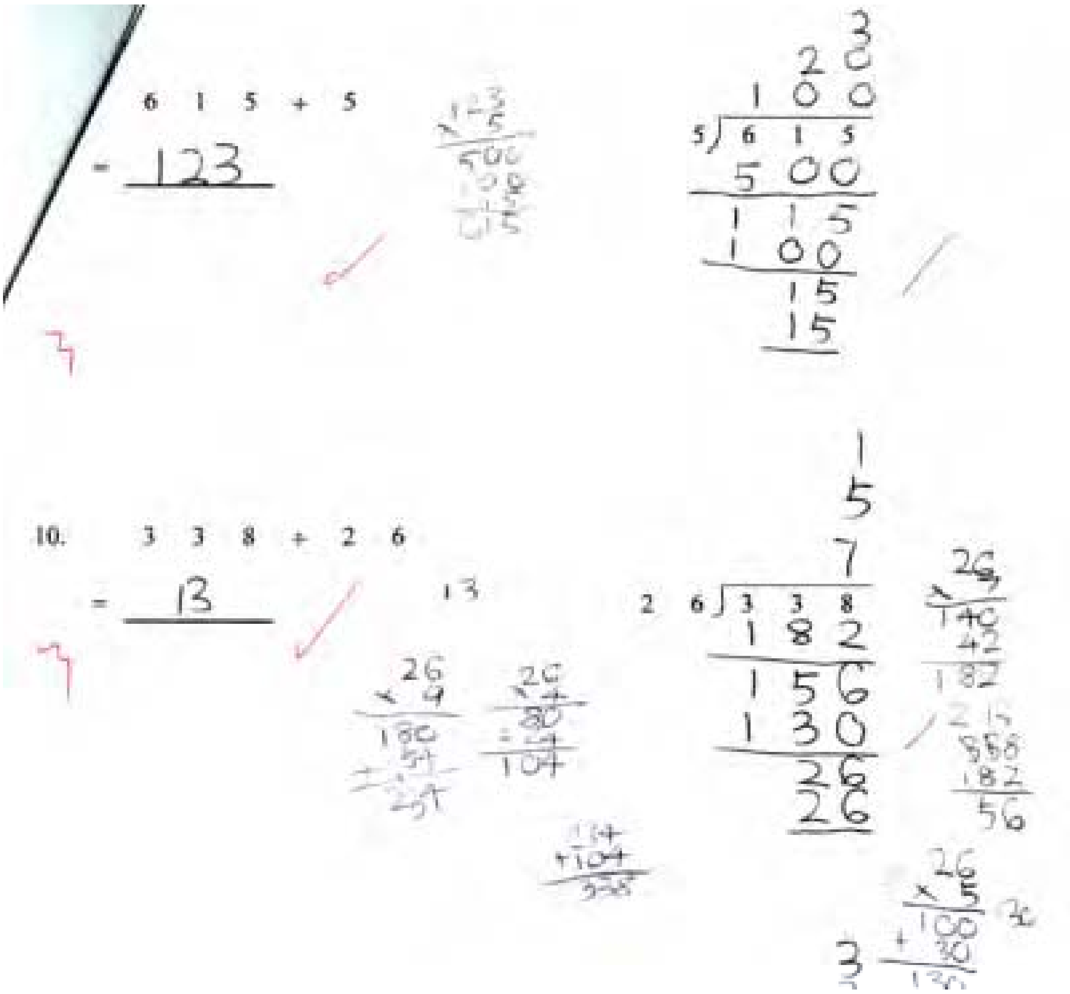
Table 2 *Control and Experimental Group Performances on the Long Division Pre- and Post-Tests*

	Experimental Group (85) [b]		Control Group (70)	
Test items [a]	M	SD	M	SD
Pre test(24)	12.84	6.16	11.89	6.72
Post-test(33)	23.23	8.63	19.4	9.41

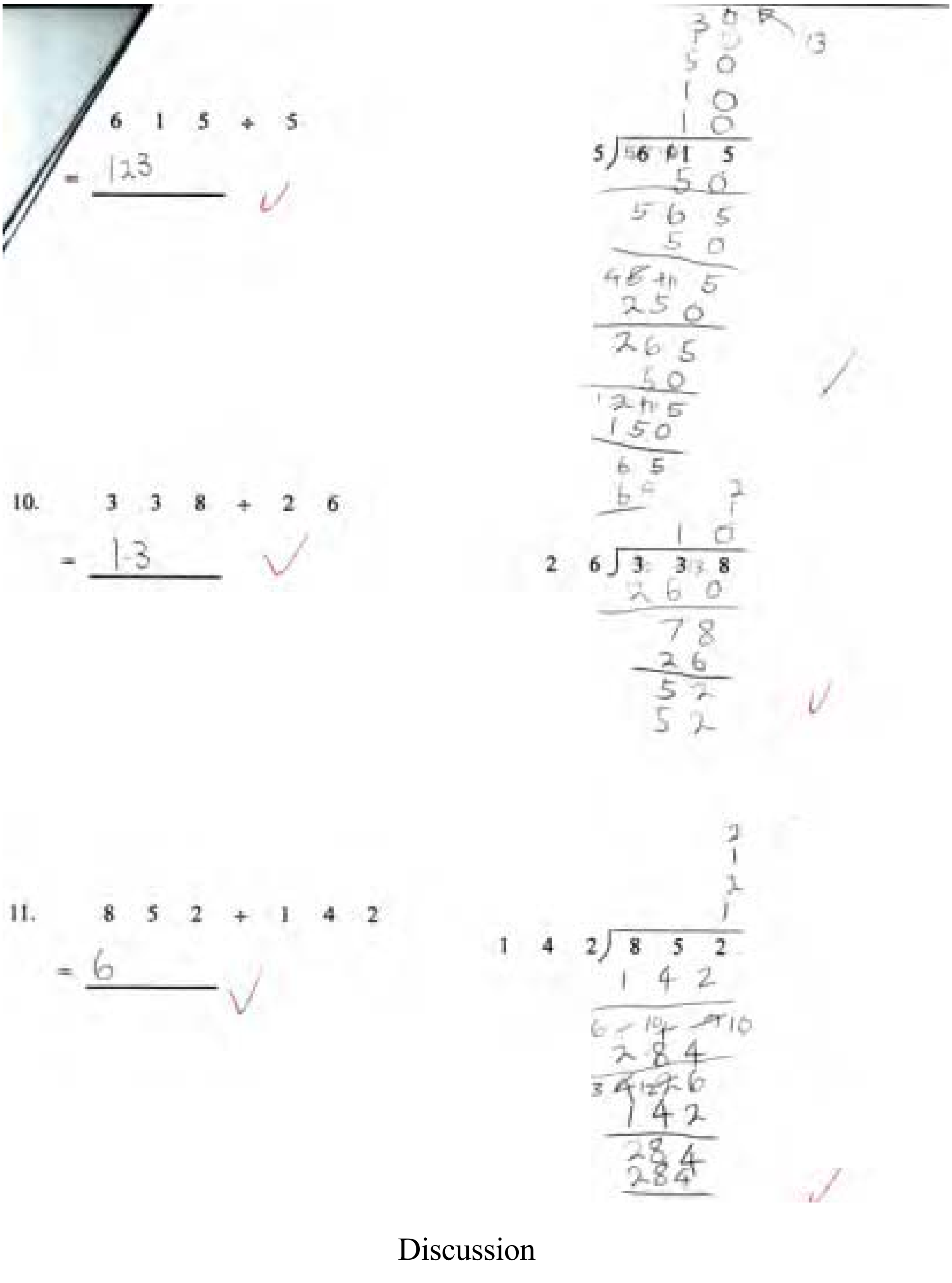
a : Total score of test in parenthesis b: Number of Students.

Some Work Examples

In this section we include examples of students' use of the POQ method. In the first problem, student A did the division in a similarly traditional way but, to skip the rules, she took the easy combination 100-20-3. Note that the intermediate remainders 115 and 15 are the results of direction subtractions. In question 10, the hardest step would be 26×7 to get 182. Again, the partition {7, 5, 1} is not unique.



In the second example, student B is clever enough to choose the elements of the partition in the multiples of 10 in question 9. After a series of operations, he easily spotted that 65 is just the multiple of 5 and 13. Therefore, the partition {10, 10, 50, 10, 30, 13} is easily obtained, and hence the quotient 123. In question 10, he basically applied the same trick to reduce the dividend from 338 to 260. Even though he could not promptly get $26\times3=78$, it was almost equally good that he could have spotted that $26\times1=26$ and $26\times2=52$. And every student would probably make no mistake to add 10, 1 and 2. In question 11, the trial partition 1 and 2 could easily produce 142 and 284 respectively. A series of repeated subtractions led to {1, 2, 1, 2}, hence the answer 6.



6 1 5 ÷ 5

= 123

10. 3 3 8 ÷ 2 6

= 13

11. 8 5 2 ÷ 1 4 2

= 6

Discussion

There are advantages in using the POQ method over the traditional method. Students need less mental effort because they do not need to guess the maximum (greatest number less than or equal to the dividend) during arithmetic. The cognitive load is high in such an optimization procedure, where finding the maximum is a necessity in the initial step of the traditional method of long division. POQ may avoid such an optimization task because it is analogous to the known sharing procedures. Based on the Law of Distribution, students may break down the division into many smaller steps for easier calculation because the magnitude of the numbers involved in multiplication and subtraction can be reduced. Furthermore, a repetition of elements of the partition is encouraged to reduce the cognitive load (e.g. $q_1 = q_2 = q_3 = 2$; or $q_1 = q_2 = q_3 = 10$ or $q_1 = q_2 = q_3 = 2 \times 10 = 20$) since the decomposition of the quotient is arbitrary and it can be attained bit by bit via a series of easier multiplications and subtractions in the process of division.

POQ is basically an analogue of the well known Euclidean Algorithm(EA) (Dudley 1978) of finding greatest common divisor (gcd) of two integers. In particular, the gcd of D and d is d itself if D is divisible by d and, in this case, the EA has only one step. In POQ, we do not impose (as Euclidean Algorithm does) the condition that the remainder (R_i) in each successful step of writing $R_i = q_{i+1}d + R_{i+1}$ (R_1 is just D) must be smaller than the divisor (d). In fact, the division process will be carried on if R_i is greater than d in the i^{th} step. This computational procedure, without imposing the condition that the remainder is smaller than the divisor in each successful step, is very much similar to the mental process of grouping and sharing in the real life situation children experienced.

Conclusion

Our preliminary result shows that students perform better in using the POQ then those using traditional method in long division. The long division algorithm is essentially a pencil-and-paper type of skill that must be mastered before learning and understanding complex mathematic skills such as operation on polynomials in algebra. We find that fewer rules are needed in the procedures of POQ and we do not need to handle the position of the respective digits in the calculation as in the traditional long division because we directly compute the partition of the quotient in POQ. Our observation is that POQ is more appealing to students as it can avoid the difficult estimate-and-match type mental process. Hence education practitioners are encouraged to teach children this alternative algorithm along with the traditional long division method.

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