Emerging Issues in the Investigation of the Construct of Partitive Quotient

Lucas McGee	Lisa Kervin	Mohan Chinnappan
University of Wollongong	*University of Wollongong*	*University of Wollongong*
lm04@uow.edu.au	lkervin@uow.edu.au	mohan@uow.edu.au

Partitive quotient construct is instrumental to the development of a deeper understanding of rational number. In this paper we examine this and the need to study the difficulties children experience whilst they attempt to solve fraction problems involving the use of the partitive quotient construct. In order to understand the concept of partitive quotient, it is necessary to examine what partitive quotient is and how it fits within the broader context of numeracy learning. This paper will review key literature sources that examine the meaning and importance of fractions in numeracy, and identify emerging issues that are relevant to the investigation partitive quotient.

Introduction

Mathematics is concerned with structures and operations with mental images that can be manipulated by young children. To process and make decisions based on the vast quantities of information available today, young children must be able to think mathematically and to access information provided in a variety of ways. Nurturing the development of mathematical ideas in young children is a crucial but challenging task for teachers.

Understanding how children learn mathematics and what they know at various stages of their development has been the driving force behind the work of many educational researchers and practitioners (for example, Gould, 2005; English & Halford, 1995; Post, Behr & Lesh, 1982; Streefland, 1991). This line of inquiry aims to understand and describe the mathematical development of children to inform research, teaching and curricula decisions of those working in the field. While our understanding of children's mathematical thinking is now quite extensive, it is by no means complete.

Fractions and Numeracy

Mathematics has a prominent role in the primary school program. It is second only to reading in the amount of time devoted to it and the amount of money spent for curricula materials. In New South Wales, the organisation of the content of the mathematics curriculum in stages identifies the knowledge, skills and understanding that are to be achieved by a typical student at the end of that nominated period (Board of Studies, 2002). Embedded in all stages and strands within the current curriculum document is the concept of numeracy development.

Numeracy can be described as the ability to effectively use mathematics required to meet the general demands of life at home and at work, and for participation in community in civic life (Board of Studies, 2002). To be numerate is to use mathematical ideas effectively to make sense of the world. A significant part of being numerate is possessing strong number sense. Students with sound number sense can easily model numbers and their properties. Number sense develops as children understand the size of numbers and the relationship between different types of numbers; including fractions.

The study of fractions is foundational in mathematics, yet it is identified to be among the most difficult topics of mathematics for primary school children.The learning of fraction concepts by young children has been the subject of study among mathematics teachers and educational researchers worldwide. There are numerous studies that explore the many reasons why students in primary schools experience difficulties when working in the domain of fraction numbers (for example, Gould, 2005; Moss & Case, 1999; Streefland, 1991). Despite the voluminous amount of literature that is available for educators on rational numbers, research suggests that teachers and students continue to experience difficulties within the domain of fractions. Errors in students' conceptual understandings of representations of fractions are often highlighted in their solutions involving operations with fractions.

Whole Numbers to Rational Numbers

The transition from reasoning with whole numbers to rational numbers continues to be an area where primary school students experience difficulty (Gould, 2005). This becomes especially evident when students are asked to work in the area that deals with operations of fractions. As example, Gould (2005) argues that primary students may reason 1/8 is larger than 1/7 because 8 is larger than 7. Therefore, the mental picture the student has of the proportional size of these rational numbers might not act so well as cross-reference for the student to check answers against in the case of operations involving the two rational numbers. Believing that 1/8 is larger than 1/7 because 8 is larger than 7 appears to be reasoning based on students continuing to use properties they learned from operating with whole numbers.

Moreover, problems with the transition of conceptual understandings from whole numbers to rational numbers experienced in primary school continue in high school. Moss and Case (1999) found in their study that the majority of Grade 9 students, when asked to estimate the sum of 11/12+7/8, chose 19 or 20 as the answer from a multiple-choice format. This seems to suggest that, in these students' earlier primary school education, a strong understanding of rational number concepts, highlighting the quantities involved in fractions, have not been developed.

Children need to achieve the necessary understanding of the quantities involved in fractions in primary school through partitioning a range of area, set and model manipulatives (Moss & Case, 1999). They will then begin to understand proportions and quantities represented by the symbolic representation of fractions used during operations. Learning experiences that involve partitioning and sharing objects assists children form correct mental pictures of the quantities involved in fractions. The partitioning of an object or a collection of objects highlights the difference in the meaning of numerals when they are used in rational number contexts.

Analogs in the Learning of Fraction

The use of an analog and what it represents needs to be related to the symbolic representation of the fraction. However, the choice of analog used is crucial in assisting children's mapping from the source to the symbol. It is important that the analogs used assist students construct a bridge between the analog and the fractional symbol (Charles, Nason & Cooper, 1999). Charles et al. (1999) state that, when used in the teaching of whole number concepts, unstructured analogs have been very successful. However, the use of unstructured analogs to facilitate the learning of fractions has not been so successful.

Moreover, when using unstructured analogs to facilitate fractional understanding, conceptual mapping between the discrete objects in the source and the target fraction concept is often complex in nature. For example, if in a collection of seven counters, there were three red and four blue counters, then 3/7 would be red and 4/7 of the counters would be blue (Charles, Nason & Cooper, 1999). For the student to construct these two symbolic representations (the fractions 3/7 & 4/7) from the concrete model (the analog/discrete object), the child would have to recognise that there were seven counters in the whole collection and therefore that each fractional part could be expressed as sevenths. Thus, when used to facilitate the teaching of fractions, unstructured analogs impose a high information-processing load on young children.

Regional models (squares, rectangles, circles) and length models (lines) form the basis of structured analogs that can be used in the teaching of fractions. According to Charles et al. (1999) structured analogs based on continuous region and length models generally are considered as being more appropriate for the initial learning of fractions as it is easier for students to see the whole being partitioned into pieces rather than pieces making up the whole such as with unstructured analogs. However, unlike using analogs with whole numbers where students can count and add more easily on their own, analog use in rational number learning presents young children with some complexities. This is because the actual analogs on their own cannot impart meaning in the same way they can when used in learning whole numbers. It is difficult for children to map from concrete (the analog) to abstract (the symbol) using analogs in fraction education and it is crucial that the accompanying mapping language and processes be appropriately modelled. Research also shows the importance of understanding the complexities children may experience when relating the analog back to the symbolic representation of the fraction (Post, Cramer, Harel, Kieran & Lesh, 1998). Building sound background knowledge of fractions needs the partitioning and sharing of unstructured analog objects. Partitioning and sharing unstructured analogs in real-life related contexts offers the learner a more authentic mathematical experience.

Beyond a Part-whole Understanding of Fractions: The Construct Theory of Rational Number

Charles and Nason (2001) argue that many of the problems with the learning of fractions can be attributed to efforts that have been focused almost exclusively on the part-whole construct of fractions. Oksuz & Middleton (2005) suggest that the restricted part-whole interpretation of rational numbers is an obstacle in algebraic reasoning. However, mathematics instruction in primary school devotes little time to developing conceptual understanding of the meaning of fractions beyond simple part-whole conceptual tasks (Cramer, Post & delMas, 2002). Researchers have questioned the wisdom of this practice (Charles and Nason, 2001). The construct theory of rational numbers suggests ways of understanding rational numbers through a variety of constructs such as the part-whole construct, the operator construct, the ratio construct, the measurement construct and the quotient partitioning construct (Behr & Harel, 1990; Post, Behr & Lesh, 1982).

Constructs

The part-whole construct of a fraction represents one or more parts of a unit that have been divided into some number of equal-sized pieces. The operator construct of rational numbers acts as mapping, taking some set or region and mapping it onto another set or region. The ratio construct compares any two quantities and their relation to one another (no partitioning of one object is needed). The measurement construct is most frequently accompanied by a number line or measuring device that starts at zero and extends through with the segment is broken into b segments. The quotient partitioning construct depends on partitioning. Partitioning is the process of dividing an object or objects into a number of disjointed and exhaustive parts. This means that the parts are not overlapping and that everything is included in one of the parts (fair shares), which represents a division (English & Halford, 1995).

A mathematically literate person in the rational number domain has an integrated view as to how the various constructs part-whole, ratio, decimal, indicated division; measure and operator interact and are related to one another demonstrating deep knowledge and understanding (Post et al., 1998). Deep knowledge in the primary school classroom can take the shape in learning experiences which are focused on a small number of key concepts and ideas within a topic, and on the relationship between and among those concepts (NSW Department of Education: Professional Support Curriculum Directorate, 2003). Deep understanding enables students to demonstrate a profound and meaningful understanding of central ideas and relationships between and among those central ideas (NSW Department of Education, 2003).

In order for children to develop deep knowledge and understanding in the area of rational numbers, they would need to be provided with tasks that encourage them to use not one but many of these constructs. This would assist them to develop competency with a small number of key concepts, rather than focusing on one in particular, such as the part-whole construct. For example, the quotient construct represents a different fraction idea for students and involves more complex mapping than the part-whole construct (English & Halford, 1995). In fact, students are required to apply their understanding of the part-whole construct to solve quotient partitioning problems. It relates heavily to informal knowledge learned during sharing situations. It has recently been noted that sharing and partitioning is the missing link in building fractional knowledge and confidence (Siemon, 2003).

Relationships Between Division and Rational Numbers

Middleton, de Silva, Toluk and Mitchell (2001) attribute difficulties that children experience when learning fractions to the fact that rational numbers take different meanings across different contexts. Fractions may be interpreted in different ways depending on the context. For example, two quarters (2/4) may be thought of as two equal parts of one whole that has been divided into four equal parts. Alternatively, two quarters (2/4) may be thought of as two equal parts of two wholes that have each been divided into quarters.

More specifically, Middleton et al. (2001) state that building from the contexts of fair sharing, under the quotient interpretation, rational numbers become quotients of whole numbers. A fraction is a part of some division unit; in many examples the unit is a quantity such as a metre, or a shape such as a rectangle whose area can be divided into parts. Therefore, fractions can be seen as a yield of a division situation and are, in fact, division situations in their own right. The authors suggest that there is a lack of study addressing the relationship between division and rational numbers and whether this traditional separation of is pedagogically sound practice. Especially, considering that partitioning and sharing objects learning experiences play an important role in assisting students form correct mental pictures of the quantities involved in fractions. Partitive quotient problems not only assist primary students understand the quantities involved in fractions but also gives them the opportunity to build and develop concepts; which are crucial to later high school mathematics problem involving algebraic fractions.

The Partitive Quotient Construct

The partitive quotient construct (quotient partitioning construct) of fractions is an area that is currently receiving attention from mathematical researchers due to its reliance on and reflection of a deeper knowledge and application than the part-whole construct. Oksuz and Middleton (2005) argue that partitive quotient understandings are the foundation for later high school mathematical experiences with algebraic fractions which take the form of quotients. The partitive quotient construct can be operationally defined as the process in which one starts with two quantities x and y, treats x as the dividend and y as the divisor and by the operation of partitive division obtains a single quantity x/y. For example, the fraction 3/4 could be seen as 3 pizzas shared among four people.

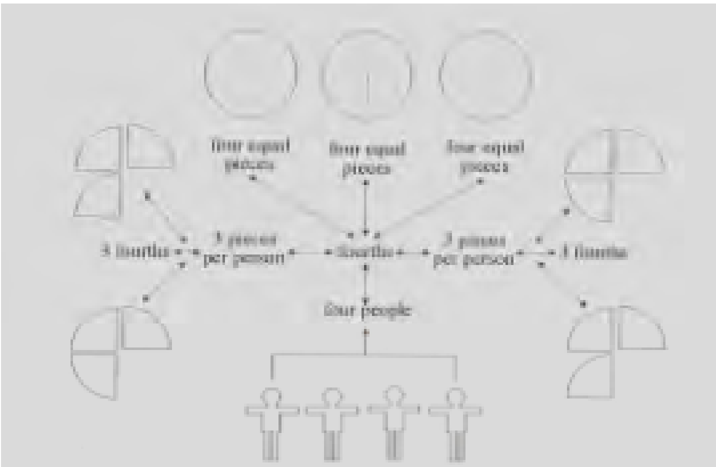

Figure 1- How three pizzas are shared among four people using partitive quotient construct? The parts shared are fractionally equal in this model of the fraction (English & Halford, 1995, p.131).

Figure 1 shows that the partitive quotient fraction construct can be readily modelled in the primary school classroom with hands-on activities using concrete materials which relate closely to many children's 'real world' experiences (Charles & Nason, 2001). The NSW Mathematics syllabus (Board of Studies, 2002) asks students to solve various complex problems using fractions. For example, students in Stage 2 are asked to model and represent fractions with denominators of 2, 4 and 8 interpreting the denominator as the number of equal parts a whole has been divided into and, interpreting the numerator as the number of equal fractional parts, such as 3/4 meaning 3 equal parts of four. In other words, students are asked to partition various shapes into equal parts to be shared. Underlying this notion is partitive quotient. By the end of Stage 2, students are expected to have constructed an understanding of various fractions involving these particular shapes.

However, there are issues surrounding the abstraction of the partitive quotient construct using concrete materials. An abstraction is a lasting change that enables the identification of the same concepts, structures and relationships in many different but structurally similar tasks. This is an important aspect and feature of deep learning and understanding. The abstraction of a concept facilitates the transfer of learning from specific contexts, such as the ones teachers set up in the classroom, to general contexts outside the classroom. For example, the abstraction of the partitive quotient construct of the fraction _ from the concrete activity of sharing three pizzas among four people requires children to construct a conceptual mapping between the number of people (4) to the fraction name of each share (fourths-as represented in Figure 1).

Moreover, this lends to the complexity of the task and the need to have a more profound and deeper understanding of rational number concepts. In the problem mentioned above, children are required to construct a conceptual mapping between the number of pizzas being shared (3) and the number of fourths in each share (3 fourths) (Charles & Nason, 2001). Such an activity provides students with an opportunity to build on rational number knowledge to the point of abstraction. In a mathematical activity, abstraction of concepts needs to occur otherwise the outcomes of tasks may be nothing more that the completion of mathematical procedures. The partitive quotient construct certainly asks students to draw on a range of complicated mathematical skills. Hence, the emphases on the importance of partitive quotient fraction construct are included in rational number learning experiences. This is highlighted by their expression of concern about the lack of knowledge currently available in regard to the abstraction of the partitive quotient understanding of fractions.

Charles and Nason (2001) have developed a taxonomy for the primary school classroom teacher. The taxonomy classifies young children's partitioning strategies in terms of their ability to facilitate the abstraction of the partitive quotient fraction construct from the concrete activity of partitioning objects. This taxonomy was based on strategies that children used during the solution of partitive quotient problems, and highlighting strategies that were most effective in generating the solutions.

However, it is important to acknowledge the role of the application of Charles and Nason's (2001) taxonomy in the planning and implementation of learning activities. Matching it with a child's level of progress towards the construction of an understanding of the partitive quotient fraction construct was not a major focus of their research study. Suggesting that investigating the efficacy of the taxonomy for evaluating a child's progress towards the abstraction of the partitive quotient fraction construct needs to be investigated in future research and teaching inquiries.

Why is it Complex for Children to Learn?

It could be argued that students experience difficulties learning fractions as it deals with concepts that are not intuitive. Thus, a link is missing between students' informal and cultural understandings and what is being asked of them to understand by a certain age in curriculum worldwide. Siemon (2003) argues that partitioning is the missing link between the intuitive fraction ideas displayed in early childhood and the more formal ideas needed to work with rational number more formally in the middle years of schooling. For example, Siemon states that children do not need to be taught how to "halve" as it is an intuitive process that most children are familiar with. Successful halving yields all of the fractions in the halving family, that is, halves, quarters and eights.

This does not suggest that from the perspective of the quotient construct, students can work out that if 2/4 being four people sharing two pizzas, they will get half a pizza each, which involves a much more complex mapping process.

Thus, research which aims to highlight the difficulties children might be having in learning about fractions may be best studied by using partitioning activities in authentic contexts (Mack, 2001).

Summary

The above review indicates that there is consensus among educators and researchers that young children need to develop a deeper understanding of fractions in order to enhance their numeracy levels.

Within the general area of fractions, the construct of partitive quotient has emerged to be an issue that is in need of further inquiry. Our review has highlighted the following issues and potential lines of inquiry:

- *Use of analogs*: Building sound background knowledge of fractions needs the partitioning and sharing of unstructured analog objects. Partitioning and sharing unstructured analogs in real-life related contexts offer the learner a more authentic mathematical experience. Research needs to examine how children model partitive quotient in authentic mathematical contexts
- *Australian context*: There is limited research that documents the difficulties Stage 2 students (those working in Grades 3 and 4) experience when mapping from the symbol to the analog when solving problems involving partitive quotient construct.
- *Abstraction*: The grasp of partitive quotient reflects a level of conceptual abstraction (Charles and Nason, 2001). How these abstractions allow children to transfer their understandings across contexts and analogs is not clear.

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