Student Thinking about Eigenvalues and Eigenvectors: Formal, Symbolic and Embodied Notions

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When entering university students often find there is a shift in presentation of mathematical ideas, from a primarily procedural or algorithmic school approach to a presentation of concepts through definitions and deductive derivation of other results. For many a course in linear algebra is the first occasion that this shift is encountered, since calculus may approximate to what they have seen at school. This research uses the theory of processes and objects, along with the ideas of embodied or visual, symbolic and formal approaches to mathematics learning to investigate some first year students' understanding of eigenvalues and eigenvectors. We identify some fundamental problems with student understanding of, and hence working with, the definition of eigenvector, as well as with some of the concepts underlying it.

Introduction

For many university students one of their first introductions to the formal presentation of mathematics is a course in linear algebra. Instead of comprising primarily manipulation of symbols in order to solve problems, the focus moves to presentations of concepts, and derivation of further concepts from these. The concepts may be presented through a definition in natural language, which may have embedded symbolism, or be linked to a symbolic presentation. These definitions are considered to be fundamental as a starting point for concept formation and deductive reasoning in advanced mathematics (Vinner, 1991; Zaslavsky & Shir, 2005). Eigenvectors form a good example of how a word definition may be immediately linked to a symbolic presentation, $Ax = \lambda x$. The definition often used is a version of:

> A non-zero vector x is called an eigenvector of a square matrix A if and only if there exists a scalar λ such that $Ax = \lambda x$.

However, eigenvectors have a strong visual, or embodied metaphorical, image in the vector space R^n, which is sometimes hidden from students by the strength of this formal and symbolic emphasis. A developing theory by Tall (2004a, b), extending some of the action, process, object, schema (APOS) ideas of Dubinsky (Dubinsky, 1991; Dubinsky, & McDonald, 2001), proposes that learners of mathematics can benefit from experiencing the results of actions in an embodied world, and processes in a symbolic world (or stages), before being able to live in the world of formal mathematics. This theoretical position suggests that it would assist university students if they were presented with embodied aspects of concepts, and associated actions, wherever possible. Extending his idea of an embodied manner of learning about differential equations (DE's) (Tall, 1998) in which an *enactive* approach builds an *embodied* notion of the solution to a DE before introducing algebraic notions, Tall (Tall, 2004a, b) has recently developed these ideas into the beginnings of a theory of the cognitive development of mathematical concepts. He describes learning taking place in three worlds: the embodied; the symbolic; and the formal. The embodied is where we make use of physical attributes of concepts, combined with our sensual experiences to build mental conceptions. The symbolic world is where the symbolic representations of concepts are acted upon, or manipulated, where it is possible to "switch effortlessly from processes to *do* mathematics, to concepts to *think* about." (Tall, 2004a, p. 30). Movement from the embodied world to the symbolic changes the focus of learning from changes in physical meaning to the properties of the symbols and the relationships between them. The formal world is where properties of objects are formalized as axioms, and learning comprises the building and proving of theorems by logical deduction from the axioms. After eigenvalues and eigenvectors are introduced to students through the formal concept definitions, they are soon into manipulations of algebraic and matrix representations, e.g. transforming $Ax = \lambda x$ to $(A - \lambda I)x = 0$, and solving this using matrices. In this way the strong visual, or embodied metaphorical, image of eigenvectors is obscured by the strength of this formal and symbolic emphasis. However, presenting an embodied approach before recourse to matrix procedures might give a feeling for what eigenvalues, and their associated eigenvectors are, and how they relate to the algebraic and matrix representations.

Such multiple representations of concepts are important since students require an ability to establish meaningful links between representational forms, referred to as *representational fluency* (Lesh, 1999). This notion forms part of *representational versatility* (Thomas & Hong, 2001; Thomas, in press), which includes a) addressing the links between representations of the same concept, b) the need for both conceptual and procedural interactions with any given representation, and c) the power of visualization in the use of representations. Such understanding is so important that it has been suggested that 'a central goal' of mathematics education should be to increase the power of students' representations (Greer & Harel, 1998, p. 22). In terms of curriculum, Moshkovitch, Schoenfeld and Arcavi (1993, p. 97) suggest that we should ask "Does any curriculum we propose make adequate connections across representations and perspectives? If not it had better be revised". One reason for this strong emphasis is that, according to Lesh (2000, p. 74), the idea of representational fluency is "at the heart of what it means to 'understand' many of the more important underlying mathematical constructs". In linear algebra too it has been recognised by Hillel (2000) and Sierpinska (2000) that conceptual difficulties are often linked to its three kinds of description or representation: the general theory; the specific theory of R^n and the geometry of n-space. Forming the links between these abstract, algebraic and geometric levels or representations is what is at the basis of many student problems and there is a need to make explicit links between them.

In this paper we use some of these ideas to analyse the way that students think about eigenvector and eigenvalue concepts, and how they cope with some cognitive obstacles. We describe how some second year university students, following a course that has little in the way of a visual component, were engaged in describing their understanding of eigenvectors, using a test containing a concept map. The results describe the understanding the students displayed, and the state of emerging links forming between parts of students' concept images of eigenvalue and eigenvector from the three worlds.

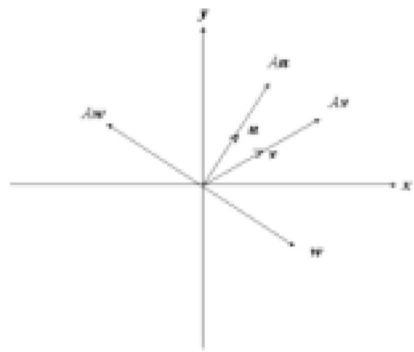
Method

This research comprised a case study of second year university students' understanding of the concept of eigenvalues and eigenvectors, and was carried out at the University of Auckland early in 2006. There were 42 from the total of 260 students in the Maths 208 course who volunteered to participated in this study by sitting a linear algebra test on the concept of eigenvalues and eigenvectors. The test was designed to examine students' geometric, matrix and algebraic understanding rather than procedural abilities. The Maths 208 course is one of the prerequisites for commerce and economics students, and is a recommended course for students with a less strong mathematics background. The course includes both advanced linear algebra and calculus, but uses a different approach from the mathematics major courses. Thus, although the course manual is filled with formal definitions and theory, and many references to Anton and Busby's (2003) "Contemporary linear algebra" textbook, it is designed in such a way that students are able to pass the course simply by knowing the routine processes, and not necessarily understanding the theory. This year, unlike previous ones, the course started with linear algebra since many students had tended to drop out of the course early on when they found sequences and series too difficult.

1. Define the notion of eigenvalues and eigenvectors in your own words.
2. Can $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ both be eigenvectors for a given matrix? Explain your answer.

3. Concepts maps are often a good way of learning about a new concept. Here is a concept map for the derivative of a function. Draw one below for eigenvectors and eigenvalues.

4. If A is a 2 by 2 matrix, explain why the picture below is not possible.



5. If $Ax = \lambda x$ put in all the necessary steps in order to show that

$$(A-\lambda I)\,x = 0.$$

6. (a) What do these all have in common? Explain.

$$\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

(b) Fill in values a,b,c,d that do not follow the above pattern. $\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$

Some formatting changed.

Figure 1. The test questions.

Results

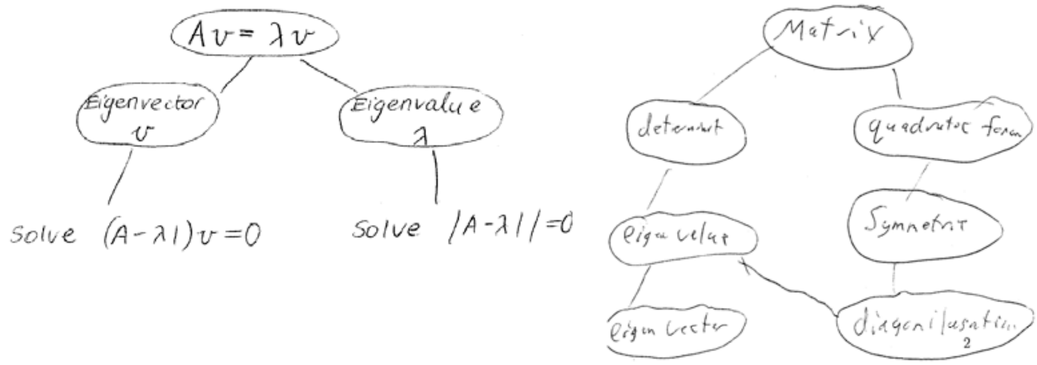
The section on eigenvectors and eigenvalues in the Maths 208 coursebook does not contain a single diagram, and thus totally ignores the embodied aspects of learning this topic. Question 4 was designed to see if the students had abstracted and assimilated to their eigenvector schema the geometric idea that when an eigenvector is multiplied by the transformation matrix it ends up in the same direction (but not necessarily the same sense). A 2x2 matrix can not have three independent vectors all with this property since it can only have at most two eigenvalues. To answer the question students had to see geometrically that each of the three vectors satisfied the eigenvector definition, link this to the data from the matrix size, and use this inter-representational reasoning to see there is a contradiction.

Of the 42 students 14 were unable to answer the question at all, and only 6 gave a correct explanation of why the diagram was not possible. These included student E, who wrote "The picture above implies A has 3 eigenvectors of different directions (but if A is 2x2, it has a maximum of 2 eigenvectors of different directions.)", student F who said "If A is 2x2 matrix, it can have maximum of 2 linearly independent vectors in its basis. Therefore one of A_w, A_u and A_v must be impossible." and student V "Diagram shows 3 eigenvalues/eigenvectors a 2x2 matrix should have only 2." Others (brackets contain the student letter) were unable to relate the picture to the concept of eigenvector, and instead a number seemed to relate it to the basis for a space, which may have been the place where they had seen a similar diagram. They wrote comments such as "Since there are only 3 vectors it will generate a space." (Q), "because you don't need that many vectors to span the plane" (U), "Maybe too many dimensions?" (AC), "⇒ linearly dependent" (AE) and "It's got way to [sic] many vectors in it." (AD). Some appeared confused and wrote, for example "The picture shows scalar multiplication which should not occur in a 2x2 matrix." (L), "Because *w* is in a different direction." (W) and "because the vectors are on different planes." (X).

We asked question 2 in order to see if the students' understanding of eigenvectors was limited to the algebraic and matrix (vector) representations or whether they used embodied, visual explanations in their answers. Of the 42 students, 14 correctly answered the question, and 4 could not write anything. However, of those who were correct 13 used only an algebraic or matrix procedural explanation, often involving multiples, such as "Yes $\begin{bmatrix} -3 \\ 4 \end{bmatrix}^{\times(-1)} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$" (T) "Yes they are merely a factor of –1 of each other" (A), "Yes, since the eigenvalues are –1 and 1" (Q) and "Yes. Eigenvectors of a given eigenvalue is any multiple of any given eigenvector." (E). Only very occasionally was a geometric comment made "Yes as $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ & $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ are multiples of each other in the opposite directions" (C). Some confusion again showed through with 7 students commenting on eigenvectors having to be independent "No, because there is a linear relationship between them." (K) and "No, eigenvectors of a matrix should be linearly independent." (L). This does not mean that the students answering in a non-geometric manner were not able to think geometrically. However, it does imply that this mode of thinking is definitely not at the forefront of their approach when the question is presented in a matrix format.

This lack of a link to a geometric perspective was certainly confirmed by the definitions in question 1 and the concept maps drawn in question 3. The first question asked students to relate their understanding of the formal definition, but without repeating it. Of the question 1 responses 16 did not write anything (or wrote 'no idea'), 17 gave a procedural response based on the equation $Ax = \lambda x$ or 'multiples' of a vector, and only two made any mention of geometric idea, either correctly stating that "A matrix 'A' when multiplied by a vector '*v*', the resulting vector has the same direction as the original vector

‘v’.”(E) or wrongly saying that they “generate a plane” (T). In addition there was one vaguely conceptual answer and 6 that we were unable to categorise. In question 3 not a single student put anything even remotely linked to geometry in their concept map for eigenvector and eigenvalue. 18 of the students did not draw anything at all, and of the remainder 21 drew a procedural map and 3 a conceptual one, with few, or no, action verbs. Figure 2 gives a typical example of the procedural concept maps, and one of the rarer concept ones. Student E sees the solving process as the only relevant detail, while student G presents only concepts, including a link to one from another part of the course.



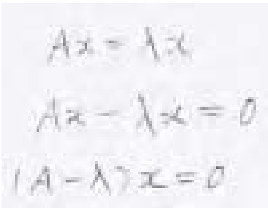
Student E Student G

Figure 2. Procedural and conceptual concept maps.

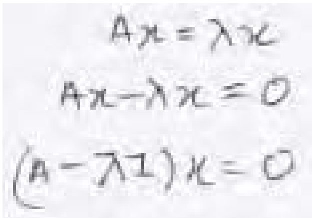
Question 5 was aimed at a possible serious process/object related problem for students with $Ax = \lambda x$, namely that the two sides of the equation are quite different processes, but have to be encapsulated to give equivalent mathematical objects, as pointed out by Stewart and Thomas (2006). In this case the left hand side is the process of multiplying (on the left) a vector (or matrix) by a matrix, while the right hand side is the process of multiplying a vector by a scalar. Yet in each case the final object is the same vector. We wondered if this process/object tension in the equation has an effect on student understanding of what is a crucial part of the definition of an eigenvector. Moreover, we noticed that the coursebook for Maths 208 glossed over the steps required to go from $Ax = \lambda x$ to $(A - \lambda I)x = 0$. Figure 3 shows a section of the coursebook where this is presented. On the surface there seems a subtle change of object from a scalar λ to a matrix λI, but the nature of I is not discussed. Beneath we see the corresponding section from the textbook, and here a small step is inserted, showing $Ax = \lambda Ix$, but it is not emphasised that it is $\lambda(Ix)$. This has the effect of changing the process on the right hand side to one very similar to that on the left, namely multiplication of a vector (or matrix) by a matrix (and then by a scalar afterwards).

Figure 3. The coursebook and textbook explanations of the move from $Ax = \lambda x$ to $(A - \lambda I)x = 0$.

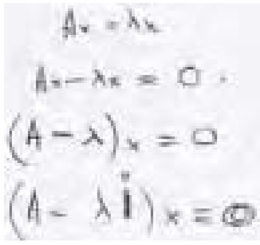
We wanted to know how the student perspective on this equation-changing would influence their ability to perform the task, and hence the question. In the event it proved to be quite revealing. It was clear that 13 of the students did not understand what the *I* was, where it came from, and why it was there. We see in Figure 4 that this affected their ability to complete the relatively simple three-line transformation of the equations. These three students, A, K and S, either ignore the identity matrix or simply insert it in the final line.



Student S



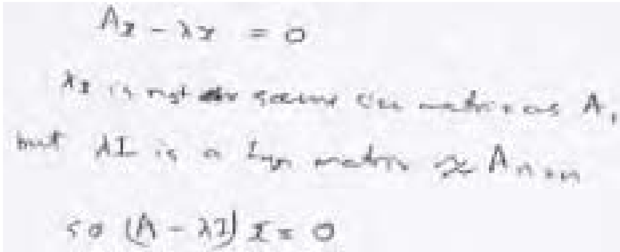
Student A



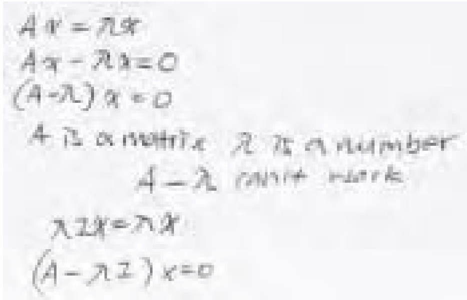
Student K

Figure 4. Working of students A, K and S on question 5.

Some evidence of what was causing the difficulty was found in the explanations of other students. Figure 5 shows the work of four more students, C, J, L and P. Here students C and J are finding it difficult to explain why the λ seems to become λI. Student J tries to explain, with little understanding, that "E[igen]-values must have Identity matrix, otherwise can not be expressed." and hence the *I* has to be inserted. However, students L and P have both decided that $A - \lambda$ cannot be accomplished ("can't work") since they are of different types—"*A* is a matrix λ is a number"—and so it is necessary to "multiply [λ] by the identity matrix" as a solution, and P almost correctly performs this. On the other hand, student C is clearly struggling with the idea that the order of λx will not be the same as that of *A*, but is happier that λI is also an *nxn* matrix. To overcome the difficulty he has focussed on the input objects on each side of the equation that are operated on, rather than the object produced by the process, and the processes are still causing cognitive conflict.



Student C

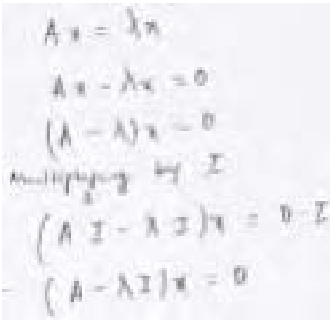


Student P

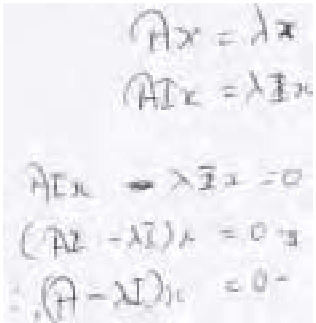
rking of students C, J, L and P on question 5.

either performed an operation that they were used to and the equation through by I (and assuming associativity) or replaced immediately by student B, but student Q, like P, e was clearly a problem with the $A–\lambda$ (see Figure 6), and On the other hand, student E chose to follow the textbook,

Student E



Student Q



Student B

Figure 6. Successful working from students B, E and Q on question 5.

Question 6 considered whether students could recognise the equation $Ax = \lambda x$ in the matrix representation, and interpret it accordingly. 23 of the students correctly linked across the representations, stating, for example, that “All the vectors are eigenvectors.” (A), “They all have eigenvalues & eigenvector” (M) “All results are multiples of x (i.e. x is an eigenvector)” (V). A further 8 could see that the resulting vector was a multiple of the original, but did not link this to eigenvectors. 7 students did not answer the question and 4 wrote an incorrect answer. This evidence suggests that students were much better at linking the two representations in the symbolic world than they were for the geometric.

Conclusion

It seems that our data show two things that are important for the teaching of eigenvectors and eigenvalues in linear algebra. Firstly, while they seem reasonably confident with the algebraic and matrix procedures, and were able to relate versions of the equation $Ax = \lambda x$ between them, the vast majority of our students had no geometric view of eigenvectors or eigenvalues, and could not reason on the relationship between a diagram and eigenvectors, to their detriment. This is not surprising since the coursebook did not present such a view, and it appears that the lecturers did not do so either. However, since embodied notions of mathematics are regularly employed at all levels of mathematical thinking it is something that should be put in place. This is in agreement with the suggestion of Harel (2000), who, while cautioning that some students persist in seeing a geometric object as the actual mathematical object and not as a representation of it, maintains that "In elementary linear algebra, there should be one world–R^n–at least during the early period of the course." (p. 185). Secondly, the progression, working within the algebraic symbolic world, from $Ax = \lambda x$ to $(A - \lambda I)x = 0$ is not perceived as straightforward by many students. We maintain that they are troubled by the two different processes in the first equation, and do not know what identity the I refers to. The textbook and coursebook tend to move the focus of attention to λI rather than Ix. Since students often seem to lack the understanding of how the second equation is obtained from the first, the implication is that this needs to be made explicit in teaching. It should be explained that the identity being used in the process is an nxn matrix, and it is the x that is being multiplied by this identity. This will also solve the process problem with the first equation, if it's done immediately.

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