High School Students' Use of Patterns and Generalisations

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This paper reports how high school students from two different schools used patterns and generalisations while working on some selected problems. The results show that the initial identification of a pattern was crucial in determining the type of symbolic generalisation, which for successful students' seemed to proceed through four sequential stages.

Generalisation is an important aspect in mathematics that permeates all branches of the subject and is a feature highlighted in the teaching of the subject at practically all levels. For example, in Arithmetic a child may generalise that multiplication of a whole number by 5 gives a product ending in 0 or 5. As a statement that is true for all members of some set of elements, theorems in geometry can be considered as generalisations (Mason, 1996). On the other hand, in algebra, we commonly use variables, which Schoenfeld and Arcavi (1988) described as general tools in the service of generalisation. So, what do we mean by generalisation?

Several attempts have been made to explain the term generalisation. Kaput (1999) claimed that generalisation involves deliberately extending the range of reasoning or communication beyond the case or cases considered by explicitly identifying and exposing commonality across the case or the cases. He added that this resulted in lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves but rather on the patterns, procedures, structures, and relations across and among them, which in turn become new, higher-level objects of reasoning or communication. This hierarchical aspect is similar to what Sfard (1991) proposed in her theory of reification, in which processes at one level become the new objects at another level. The idea of creating new objects for subsequent actions was also used by Davidov (1972/1990) who described generalisation as "inseparably linked to the operation of abstracting" (p. 13). The link between generalisation and abstraction was also highlighted by Dreyfus (1991). However, Dreyfus used the term generalisation as the recognition of some general characteristics in a set of mental objects and claimed that generalisation involves the expansion of an individual's knowledge structure. Regarding cognitive activities involved in generalising, Harel and Tall (1991) made a distinction between three types of generalisations, (a) expansive generalisation – one that extends the students' existing structure without requiring changes in current ideas; (b) reconstructive generalisation – one that requires the reconstruction of the existing cognitive structure; and (c) disjunctive generalisation – one which adjoins the new particular as an extra case or generates a new structure distinct from the others.

On the other hand, Radford (1996) claimed that a goal in generalising geometric-numeric patterns is to obtain a new result. This new result depends on the observer's conceptualisation of the mathematical objects and the relations involved. Radford added that accordingly, generalisation is not a concept but rather a procedure and that a generalisation procedure g arrives at a conclusion α, starting from a sequence of "observed facts", $a_1, a_2, \ldots, a_n$, which can be written as: $a_1, a_2, \ldots, a_n \rightarrow \alpha$ (α is derived from $a_1, a_2, \ldots, a_n$). The most significant aspect of the generalisation is its logical nature that makes possible the conclusion α.

It should be noted that inductive reasoning, which is commonly used in generalising from patterns, does not necessarily lead to valid conclusions. If there are flaws in the logic then certainly the generalisation would not be valid, and so generalisation as a didactic strategy cannot avoid the question of validity. Burton (1984) claimed that to become robust a generalisation must be tested until it is convincing so that it moves from being personal to public. Burton also mentioned that both inductive learning and deductive learning involve generalising activities. Her view is that inductive learning involves specialising, conjecturing, and generalising in that order, which is the reverse order for deductive learning.

Although generalisation may seem to be omnipresent in school mathematics, there are pedagogical issues that cannot be ignored. In her research, Lee (1996) found that generalising activities led to three types of conceptual obstacles. First, there were obstacles at the perceptual level, which concerned with seeing the actual pattern. Second, there were obstacles at the verbalising level, which involved expressing the pattern clearly. Third, there were obstacles at the symbolisation level, for example using a variable n in a general expression. Thus, generalisation in school mathematics is a very important aspect that needs to be carefully investigated. Accordingly, this study focused on how secondary students used patterns to help them generalise and what were some of their related conceptual difficulties?

Methodology

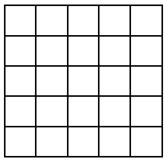
The study reported in this paper is part of a larger study investigating students' use of algebraic thinking in geometrical contexts. The study took place in two large Midwestern high schools in the United States. One geometry class was selected from each of the two high schools: School X and School Y. There were 21 students in the class from School X and 18 students in the class from School Y. The two classes were observed for three months and twelve lessons from each class were videotaped. Three students were selected from each of the two classes based on the results of an algebra test, which was developed in conjunction with the classroom teachers of these two classes and three other experts in the field. Andy, Betty, and Melanie were the focus students from School X whereas Pete, Kristina, and Abby were from School Y (all names are pseudonyms). Andy and Kristina were more able students whereas Betty and Melanie were weaker students from the sample. Each of the six students was interviewed four times for about 40 minutes each time. The interviews were audiotape recorded and then transcribed. The students were asked to solve some problems involving certain aspects of patterns and generalisations. The problems were asked sequentially, in the different interview sessions, as given in the Results section below. The questions were read out to the students and additionally a written version was provided to them. Seven problems that involved some aspect of generalisation in a geometrical context were used with the students. The problems were selected based on the topic coverage in the selected classrooms. Problem 1 has been adapted from the one by Swafford and Langrall (2000) and Problem 7 from the one used by Krutetskii (1976).

Results

In this section, the focus students' generalisation approach in the context of the seven problems is discussed sequentially. The results for the students' performance show some interesting features.

Problem 1

> How many small squares are there in the border of this 5×5 square (square drawn on a rectangular grid)? How many are there in a 6×6 square? How many are there in a 10×10 square? How many would there be in a square of side $n \times n$? If there are 76 border squares in square grid, what is the size of the grid?



In this problem the square grid provides a geometrical context for an algebraic generalisation. The three students from school X used different strategies to find the number of border squares for the 5×5 square grid. Andy did it mentally and later explained that he added 3 + 3 + 5 + 5 to get 16. Betty counted the squares one by one and then wrote 5, 3, 5, and 3 along the border of the grid. This showed that her strategy of using 5 + 3 + 5 + 3 was somewhat similar to Andy's. Melanie responded very quickly that the answer was 20, which was incorrect. When asked to check the answer by actual counting, she was puzzled that it was 16. She did not show any strategy for getting the answer other than by counting.

For a 6×6 grid, Andy did not follow his strategy from the previous part. He said the answer was 25 and added that for a 10×10 it was 81. For an $n \times n$ grid he said it was $(n\text{-}1)^2$. This clearly showed that Andy was not using his previous strategy. He did not mention why he chose $(n\text{-}1)^2$, but it seems that he was mislead by the number of border squares in the 5×5 grid as also being $(5\text{-}1)^2$. On prompting, he changed his answer and was able to come up with the correct generalisation of $4(n\text{-}1)$. He was able to use this formula for the inverse problem to find the size of the grid for which the number of border squares was 76. Betty stuck with her strategy and had no problem getting the answer for a 6×6 or 10×10 grid. She was eventually able to write down the answer for an $n \times n$ grid. She wrote $N + N + (N\text{-}2 + N\text{-}2) = 4N\text{-}4$. Betty was not concerned about the use of N instead of n in the expression. She needed some prompts to be able to set up an equation and solve it to get the size of the grid for which the number of border squares was 76. Melanie could not follow through to get the answer for a 6×6 grid. She thought that it might be 16 + 6 = 22. That is, she thought of adding one additional row of 6 squares to the previous answer of 16 for a 5×5 grid. She could not get to a 7×7 or 10×10 grid. She said she could not do it without a diagram. After several prompts, she was able to finally generalise to $4n\text{-}4$ for an $n \times n$ grid. However, for the inverse problem, she could not get the size of the grid for which the number of border squares was 76.

From school Y, Pete started this problem by actually counting the number of squares in the 5×5 grid. Since no diagram was given for a 6×6 grid, he knew that he had to be more systematic. His revised strategy was to add 5 + 5 + 6 for the 5×5 grid, thinking of the 6 as 3 + 3. He used the same strategy for a 6×6, 10×10, and also for the general case $n \times n$. For this latter case, he wrote $2n + (n\text{-}2)\times 2$, which he simplified to $4n\text{-}4$. For getting the size of the grid for 76 border squares, he wrote 76 + 4 = 80, then he wrote 80/4 = 20, to say that the size of the grid was 20×20. Kristina and Abby were able to get all of the answers and they had very similar strategies for getting the generalised value of $4n\text{-}4$ for the $n \times n$ grid. However, for finding the size of grid with 76 border squares, Kristina just substituted 20 for n to get the answer. This suggested a more trial and error strategy, whereas Abby actually set up an equation and solved for n.

Problem 2

> What is the sum of the interior angles in a triangle? From any vertex, we can divide a quadrilateral into two triangles. What is the sum of the interior angles in a quadrilateral? What is it for a pentagon, hexagon, and a decagon? What would it be for a polygon with n sides?

In this problem, the students had to know the angle sum of a triangle and the names of the polygons up to ten sides. The algebraic skills included the identification of a pattern and subsequently writing down the generalisation from the pattern. All of the focus students except Melanie were able to find the sum of the interior angles in a quadrilateral, pentagon, and hexagon by actually drawing such a figure and then counting the number of triangles they could get by drawing diagonals from one vertex. They had no problem generalising to a polygon with ten sides, even though a diagram was not used. Eventually all of them, except Melanie, were able to get the result that for a polygon with n sides the sum of the interior angles is $(n\text{-}2)\times 180°$. Melanie needed some help with the pentagon and hexagon before writing down the angle sum. For a decagon she did not do any calculation but used an additive strategy and counted on from a hexagon, which implied that she had noted a pattern in her responses, but was not quite able to formalise it. To get the result n-2) $\times$ 180 for a polygon of n sides, a table of values for number of sides and the corresponding angle sum was drawn for her. It was only when this scaffolding was done that she was able to follow the pattern and come up with the generalisation. It seemed that the organisation of the results in a tabular form was important for Melanie in triggering the identification of the pattern.

Problem 3

> What is the relationship between an interior and an exterior angle of a triangle? How many pairs of interior and exterior angles do you have in a triangle? What is the sum of all of the interior and exterior angles of a triangle? What is the sum of the exterior angles of a triangle? What is the sum of all of the exterior angles in a quadrilateral? A pentagon? A hexagon? A polygon with n sides?

The main geometrical concepts in this problem are that of interior and exterior angles. The students had to understand that the sum of an interior and the corresponding exterior angles in a polygon is 180°; and that if they knew the sum of all of the interior angles in a polygon then the sum of the exterior angles could be found by subtracting the sum of the interior angles from the sum of all of the interior plus exterior pairs. The students should then have been able to generalise from this result.

Andy was able to follow the argument and he got the sum of the exterior angles of a triangle, a quadrilateral, and a pentagon easily. He was even able to do it for a decagon and although he guessed that the answer had to be 360° for any polygon, he actually did the calculation for a polygon with n sides to confirm his guess. Betty was able to do it for a triangle, quadrilateral, and for a decagon as well. Although she guessed that the result should be 360° for any polygon, she was not actually able to do the calculations to justify the result for a polygon with n sides. Melanie, on the other hand, had some difficulties following the argument even for a triangle. After some prompting, she was able to do it for a quadrilateral and a pentagon but not for a decagon. However, she guessed that the sum of the exterior angles might always be 360° for any polygon. She was not able to do the actual calculation to justify the answer.

All of the three students from school Y were able to follow the arguments and were able to get the exterior angle sum for a triangle, quadrilateral, pentagon, and the decagon easily. They guessed early that the sum of the exterior angles in any polygon would be 360°. They all were able to do the calculation for a polygon of n sides to show that the sum of the exterior angles did not depend on the number of sides of the polygon and that it was always 360°. While checking for understanding, it was noted that the students had difficulty in applying their knowledge about the sum of the exterior angles to find the number of sides of a regular polygon if the size of one exterior angle was known. Although the students knew what a regular polygon was, none of them was able to solve such a problem.

Problem 4

> The equation of a line is $y = 3x + 5$. Write down the equation of another line having the same slope as the given line. What would be the general form of the equation of a line having the same slope as the given line?

This problem refers to the equation of a line in the slope-intercept form. The students were expected to know that in coordinate geometry, the equations of lines having the same slope varied only in the value of the intercepts. All of the six focus students were able to identify the slope of the line as 3. More specifically, Andy, Betty, and Abby wrote 3/1 for the slope. This seemed to be a common practice for writing down the slope of a line from its equation in the slope-intercept form. However, for the general form of a line having the same slope, different answers were obtained. Andy wrote $y = 3x +$ *anything*, then wrote $y = 3x + z$, where z is a number. Betty wrote $y = 3x +$ *number on y-intercept*, and Melanie wrote $y = 3x +$ *anything*. From school Y, Pete wrote $y = 3x + n$, where n is a number. Kristina wrote $y = 3x +$ *something*, whereas Abby was not able to come up with a general form for such a line. In this context where the symbol for the parameter was not provided, students found it difficult to generalise using their own symbols.

Problem 5

> All points on a circle are equidistant from its center. If P(x, y) is a point on a circle having center at the origin and radius 5, what relation can you write connecting x and y? What would be the relation if the radius was 10? What would it be if the radius was r?

To solve this problem, the students were provided with a diagram and the formula to find the distance when the coordinates of two points were given. The students also needed some algebraic skills in the manipulation of the relation that they had to write connecting x and y. All of the focus students were able to write down the relation connecting x and y using the distance formula and even the relation for the general case when the radius was given as r. All of them wrote $5 = \sqrt{[(x\text{-}0)^2 + (y\text{-}0)^2]}$, except Melanie who reversed the order in which she used the points in the formula, which was, of course, correct. Melanie wrote $5 = \sqrt{[(0\text{-}x)^2 + (0\text{-}y)^2]}$ and then she was not sure how to simplify $5 = \sqrt{[(\text{-}x)^2 + (\text{-}y)^2]}$. She even thought that $(-x)^2 \neq x^2$ and $(\text{-}y)^2 \neq y^2$. However, she was later convinced that this could be written as $x^2 + y^2 = 25$. It was interesting to note that four of the focus students Andy, Pete, Kristina, and Abby made the same algebraic mistake when trying to simplify the expression $5 = \sqrt{[(x\text{-}0)^2 + (y\text{-}0)^2]}$. They wrote $5 = \sqrt{[(x^2 + y^2]}$, but then they went on to write $5 = x + y$ and eventually wrote $x^2 + y^2 = 25$. There seemed to exist some underlying misconceptions about algebraic simplifications.

Problem 6

Two parallel lines are labeled l and m. On line l one point A is marked and on line m three points B, C, and D are marked. How many different triangles can be formed by joining three of the given four points? If the point on line l is kept fixed but one more point is added on line m, how many triangles can be formed in the same way? Can you find out the number of triangles that can be formed under the same conditions if there were 6 points, 10 points, n points on line m?

No. of points on line m	3	4	5	6	10	n
No. of triangles						

A diagram showing the parallel lines l and m and the points A, B, C, and D was given. In this problem the students had to count the number of triangles systematically as the number of points on line m was increased and then they had to come up with some rule for finding the number of triangles in the general case when there were n points on line m.

Andy and the three students from school Y, Pete, Kristina, and Abby, had no difficulty in counting the number of triangles up to $n = 6$. They had a systematic strategy and were then able to extend the result to $n = 10$, without doing any actual calculation, by just following the pattern of numbers they had obtained in the table. However, they could not come up with a general formula for the case when there were n points on line m. Only Andy came up with a recursive formula. He wrote $X + n\text{-}1$ for the number of triangles when there were n points on line m, where X for him represented the previous number of triangles. Both Betty and Melanie were not systematic in their counting of the triangles and so had difficulties in completing the table. Melanie had even more difficulties than Betty. However, once they were able to get the values in the table up to $n = 6$ after some very careful counting and some help, they were both able to identify the pattern and were able to write down the number of triangles for $n = 7$ and $n = 8$ without using a diagram.

Problem 7

Two of the sides of an isosceles triangle have measures 4 inches and 10 inches. What would be its perimeter? Why? A triangle has sides of lengths a, b, and c. What relation(s) can you write connecting a, b, and c?

This problem required knowledge about an isosceles triangle and about the geometrical fact that in any triangle the sum of any two sides is always greater than the third side. The students had to identify this geometrical fact in the first part of the problem and then to generalise it in the second part. The three students from school Y initially thought that there were two answers for the first part namely, 24 and 18. However, they soon realised that 18 was not a possibility and so gave the correct answer as 24. They were able to generalise to any triangle and wrote the relations $a + b > c$, $b + c > a$, $a + c > b$. The only difference in their answers was the inconsistent use of capital and small letters for the length of the sides. Abby used all small letters a, b, and c, but Pete used only capital letters A, B, and C whereas Kristina used a combination of both small and capital letters.

The students from school X had different responses. Andy initially thought the answer was 24 for the first part but then thought that 18 was also possible. It was only after some prompts that he finally realised the impossibility of having 18 as a perimeter. He could not give a general rule for a triangle with sides a, b, and c. However, he did mention that at least one of a or b had to be greater than half of c. This was obviously incorrect, but it seemed that his belief was that "half of a plus half of b" had to be greater than c, rather than "a plus b" was greater than c. Betty was not able to follow the first part of the problem. It was only after some help that she could do so. She wrote $A + B > C$, for the second part and with some further prompts was able to write $c + a > b$ and $b + c > a$. Melanie initially wrote 24 and 18 as an answer for the first part. She thought that both of these answers were possible. After a triangle was drawn for her to illustrate the situation, she understood that 18 was not possible. She knew that a triangle with sides of lengths 2, 3, and 7 units was not possible but she could not generalise this result to a triangle with sides a, b, and c. When the relation $a + b > c$ was written down, she was able to write out $b + c > a$ and $a + c > b$.

Discussion

In Problems 1 to 7 the focus students had to identify a general pattern starting from few specific cases. It was expected that reasoning inductively from a few cases the focus students would be able to generate a general rule or formula. Successful strategies seemed to proceed through the following sequence of stages: a direct modelling stage, the stage of identification of a pattern, the stage of proof testing of the pattern, and the final stage for finding a rule for the general case.

The direct modelling stage involved the focus students actually using strategies such as counting, drawing, or writing down the first few cases systematically. For example, in Problem 1 most of the students counted the number of squares in the 5×5 grid and some of them drew a 6×6 grid and again counted the number of squares before identifying any pattern. In Problem 2, at this stage, the students used the drawing of a quadrilateral, a pentagon, and a hexagon to find the angle sum by drawing inside those figures a certain number of triangles from a given vertex. In Problem 3, the students used the drawing for a quadrilateral, a pentagon, and a hexagon to arrive at a pattern of results for the sum of the exterior angles. In Problem 6, the students counted the number of triangles when there were 3, 4, 5, and 6 points on line m. Thus, in most of the problems the students were doing some direct modelling at this stage.

The second stage was the stage during which the students were actually able to identify some useful pattern. Which pattern one chooses depends on the particular aspect of the pattern that one wishes to observe (Phillips, 1993), and this depended considerably on the students' systematic counting, drawing, or writing/recording from the first stage. For example, in Problem 1 for the 5×5 grid, some students identified the pattern as $5 + 5 + 3 + 3$, and some as $5 + 5 + 6$. Although the two representations do not look very different, they led to slightly different ways of writing the general expression. The generalisation was $n + n + n\text{-}2 + n\text{-}2$ for the first pattern and $n + n + 2\times(n\text{-}2)$ for the second one, which was later simplified to $4n\text{-}4$. Thus a systematic way of counting the number of squares helped the students to generalise. The generalisation was fairly easy when there were sufficient examples to make the pattern quite evident. In Problems 2 and 3, the successful students were able to identify a connection between the number of sides in a polygon and the sum of the interior/exterior angles in the polygon. The systematic way of recording the number of triangles in Problem 6 in a table helped the successful students to identify a pattern in the results. In problems where this was not the case, the students had more difficulties in coming up with a useful pattern. For example, in Problem 7 the students had to come up with a generalisation based on only one initial case. This proved to be hard for the students. Lee (1996) has pointed out that the problem for many students is not the inability to see a pattern but the inability to see an algebraically useful pattern.

In the third stage, the successful focus students tested their conjectures about the patterns by using a particular case beyond the range for them to model directly. For example, in Problem 1 the students were asked to find the number of border squares in a 10×10 grid. They knew that it was not worthwhile to draw a 10×10 grid and then to count the squares one by one. Generally, the students who were able to attain this stage were able to get to the algebraic generalisation later. "Counting on" was a common strategy for some of the focus students to reach a solution for the 10×10 grid, but this was not very helpful as an overall strategy. It was when these students were asked about larger grids such as 100×100 where counting on strategies were not very practical that these students looked for alternative strategies. So, they used their earlier patterns such as $3 + 3 + 5 + 5$ or $5 + 5 + 6$ from the earlier parts to get the answer. In problems 2 and 3, the students were asked to find the sum of the interior/exterior angles in a polygon with ten sides. The students knew that it was not necessary to draw the decagon and had to rely on their previous sequence of results. Similarly, in Problem 6 the students did not put 10 points on line m to come up with the number of triangles for this case. They used the patterns they had identified to do so.

In the final stage, the students had to come up with a generalisation. Swafford and Langrall (2000) had claimed that the generalisation of a problem situation might be presented verbally or symbolically. In the problems that were used in this study, the focus students avoided a verbal generalisation and all of them tried to give symbolic generalisations. For the symbolic, this involved constructing an algebraic relation for the pattern they had noticed. Their success in the first three stages of the solution process helped them to come to the right conclusion. The students used the pattern that they had identified earlier to come up with the generalisation. For example, Betty from school X wrote $N + N + (N\text{-}2) + (N\text{-}2)$ which was similar to her $5 + 5 + 3 + 3$ strategy for the 5×5 grid. She overlooked the fact that the grid was $n\times n$ and not $N\times N$, but this minor detail did not seem to bother her. In very much the same way, the students from school Y wrote $2n + 2\times(n\text{-}2)$, following their pattern $5 + 5 + 2\times3$ for the 5×5 grid. In Problem 2, the successful students had no difficulty in coming up with the generalisation $(n\text{-}2)\times180°$ for the interior angle sum of a polygon of n sides. The pattern of results noted from the triangle, quadrilateral, pentagon, and hexagon was essential. By the time they had to find the sum of the interior angles of a 10-sided figure, they already had the pattern for the general case. It was a similar situation in Problem 3, except that the weaker students could only guess that the exterior angle sum would be 360°, but they were not able to justify it. The more successful students were able to show by subtraction of the sum of the interior angles of a polygon of n sides from the sum of all the interior and exterior angle pairs of the polygon that the result came out to be 360°.

Some of the difficulties encountered by the students, such as producing variables on their own, and writing down the relations algebraically, hampered the students' progress. For instance, the students found it very difficult to come up with a symbolic generalisation for Problem 6. Generally, the students were able to fill up the table, but their search was for a linear symbolic relationship. Most of them were able to identify a recursive relationship in the table but only Andy, from school X, gave an explicit recursive formula. His formula was $X + n\text{-}1$, where X stood for the number of triangles from the previous value of n, the number of points on line m. However, he was unable to give an explicit symbolic representation of the non-linear generalisation in Problem 6. Some authors caution that, in their attempt to write symbolic representations, students often focus on inappropriate aspects of a number pattern – particularly the recursive relationship between successive terms in a sequence (MacGregor & Stacey, 1993; Orton & Orton, 1994). Thus, in Problem 6, it might be possible that the students' focus on the recursive relationship was responsible for their inability to produce an explicit generalisation. Even Problem 4 was problematic for some of the students. In Problem 7, the students had difficulties in coming up with the generalisation about the sides of the triangle mainly because a single case illustrated the problem. It seemed that a limited number of initial cases might not be enough for the students to find a pattern and hence a generalisation from the pattern, although Dreyfus (1991) had claimed that sometimes it is better to abstract from a single case.

The three types of conceptual obstacles in generalising activities that Lee (1996) found in her research were also noticed in this study. First, there were obstacles at the perceptual level. For example, Melanie had this obstacle in Problems 1, 2, and 3. She was unable to identify the pattern and this led to her not being able to proceed further on her own. At the perceptual level, the focus students found it easy to identify patterns that showed constant differences between successive terms but not when the pattern was different. The symbolic expressions for the generalisation was obtained easily when constant differences were involved but not in problems where this was not the case as in Problem 6. Second, there were obstacles at the verbalising level. For example, Melanie in Problem 1 was not able to verbalise a useful pattern and this probably led to her incorrect generalisation. Third, there were obstacles at the symbolisation level. For example, in Problem 6 most of the focus students could not come up with a generalisation using appropriate symbols, even when they had identified a pattern. As noted by Lee in her research, the major problem for students was not in seeing a pattern, but in perceiving an algebraically useful pattern. It is important to note that some of the focus students did not verify whether the formula they had generated worked in the simplest of cases. They were generally confident that they had the right symbolic form of the generalisation. Also it is worth noting that when the students were not systematic in their recording of the results then they were unable to identify any patterns and this led them to inappropriate conclusions.

To check for understanding, the students were asked to solve the inverse problems in Problems 1 and 3. In Problem 1 students were asked to find the size of a grid for which the number of border squares was 76. Solving an equation was the most common strategy. Some of the focus students needed prompts to be able to do so. Kristina used a trial and error strategy. In Problem 3, the students were asked to find the number of sides of a regular polygon with a given exterior angle. None of the focus students were able to solve such a problem. Thus, the students seemed to have a loose understanding of the generalisations that they had come up with in the problems.

To conclude, the study shows that the identification of a useful pattern by the students was a significant factor in their successful symbolic generalisation, which seemed to proceed in four sequential stages. However, the students had difficulties with non-linear symbolic generalisations. The students generally avoided verbalising their generalisations. Students with a weaker background in algebra, such as Melanie and Betty, had more difficulties generalising compared to the other students. Even the students with a stronger background in algebra displayed some misconceptions in handling algebraic expressions. In this study, all of the problems had some connections to geometry, which may have added to the students' difficulties. In future studies, a broader range of problems with similar generalising activities may provide a more complete picture.

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