Utilising the Rasch Model to Gain Insight into Students' Understandings of Class Inclusion Concepts in Geometry

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This study extends research into the van Hiele Theory by narrowing the microscopic lens and providing a focused analysis on the understanding and development of class inclusion concepts in Geometry. This paper integrates two qualitative frameworks, identified through the utilisation of the SOLO model, that indicate developmental growth in understanding of relationships among figures, and relationships among properties. This is considered via a quantitative approach, using a Rasch analysis model, which provides a comparison of the complexity of seven different interview tasks within the context of triangles and quadrilaterals.

This study is part of a larger study that extends research into the van Hiele Theory by narrowing the microscopic lens and providing a focused analysis on the understanding and development of class inclusion concepts in Geometry. Pertinent to this study, the level associated with a student who accepts and utilises notions of class inclusion is described as Level 3 (van Hiele, 1986). This aspect of Level 3 is regarded as both a difficult concept to acquire and a prerequisite for formal deductive reasoning (De Villiers, 1998; Heinze, 2002). The networks of relations, which are the students' focus when exhibiting Level 3 thinking, can be described as those that deal with the relationships among properties within figures, and relationships among figures (van Hiele, 1986). In an attempt to refine the characteristics of the development of this concept, an initial qualitative study (Currie & Pegg, 1998; Serow, 2006) utilised the SOLO (Structure of the Observed Learning Outcomes) model (Biggs & Collis, 1982) to provide deeper insights into the van Hiele levels. A central finding of this initial study was the identification of two frameworks that describe developmental pathways leading to a) an understanding of relationships among figures, and, b) an understanding of relationships among properties. This study is a quantitative analysis of the results using a Rasch analysis model with the aim of providing further insights into students' understandings of class inclusion. Rasch measurement has been described as permitting "the identification and examination of developmental pathways, such as those inherent in the development of mathematical concepts" (Callingham & Bond, 2006).

Background

This study provides a quantitative synthesis of the developmental pathways described in Table 1, based upon the application of ACER's QUEST analysis program, using the partial credit modelling process, provided by Masters (1982). This analysis program enabled the plotting of item difficulties for the seven tasks upon a single scale and provides some initial insights into a comparison of item/category difficulty concerning tasks that target geometrical relationships within the contexts of triangles and quadrilaterals.

In addition to the van Hiele Theory, the SOLO model was utilised in the initial qualitative study. This model is comprised of two main components, these being: the modes of functioning, and, the cycles of levels. There are two modes of functioning relevant to this paper, namely, concrete symbolic (CS) and formal (F). The concrete symbolic mode involves the application and use of a system of symbols, for example, written language and number problems, which can be related to real world experiences. The formal mode is characterised by a focus upon an abstract system, based upon principles, in which concepts are imbedded. Within each mode development occurs described in terms of levels. General descriptions of the levels are the following.

1. Unistructural (U): response is characterised by a focus on a single aspect of the problem/task.
2. Multistructural (M): response is characterised by a focus on more than one independent aspect of the problem/task.
3. Relational (R): response is characterised by a focus on the integration of the components of the problem/task.

Studies (Campbell, Watson, & Collis, 1992) have extended the SOLO model through the suggestion that more than one cycle of levels exist within each mode. As a result, two cycles of levels in the concrete symbolic mode have been identified. The pathways that were identified in the earlier qualitative study were characterised by two cycles of responses of the concrete symbolic mode (SOLO), and two cycles of responses of the formal mode (SOLO).

In the initial qualitative study, the developmental frameworks that emerged through the application of the SOLO model are detailed in Table 1 below. Descriptors of the tasks used, within the contexts of triangles and quadrilaterals, are outlined in Table 2 in the Methodology section.

Table 1
Developmental Frameworks Concerning Relationships Among Properties and Relationships Among Figures

Coding	Properties	Figures
R_1(CS)	The focus of the task is upon the figure in question from which all known properties are derived. A specific example of the figure is utilised from which each property is determined. There is a strong reliance on ikonic support. The properties are perceived as features.	A single property or feature is identified to link the figures. The focus of the response is upon the identification of an observed single quantifiable aspect, which places figures into spontaneous groups. There is a strong reliance on visual cues.
U_2(CS)	The reference for the response is the figure in question. The figure determines a single property. Minimisation is understood to be "less" and is based upon the uniqueness of a single property to the figure.	Classes of figures are known by name and are characterised by a single property. The class represents an identifiable unit. Links do not exist between classes, unless supported by visual cues. Observed differences play a significant role.
M_2(CS)	The single reference remains the figure in question. The figure determines two or more unique properties, which are utilised to represent the figure. Properties remain in isolation. Minimisation is understood to be "less".	M_2 responses incorporate classes of figures, which are known by name. These classes are characterised by more than one property. Links are not made between classes where differences in properties are accentuated by visual differences.
R_2(CS) response	The focus of the response is upon a link or ordering between a pair of properties, or a pair of figures within the same context. The link is characterised by a single dominant property that precludes the utilisation of a relationship in both directions.	Relationships exist between classes of figures, which are based upon similar properties. Inclusive language is used to describe the classes of figures; hence, property descriptions allow for similarities to be acknowledged.
U_1(F) response	This type of response incorporates a relationship between two properties, or between	When prompted, tentative statements are made concerning the possibility of subsets

	two figures, and is justified accurately. Two properties are now perceived to work together, and as a result determine the figure. This single relationship has become a workable unit.	within a class of figures. There is no acceptance of this notion, however, it is able to be discussed tentatively.
M_1(F)	The response is based upon the existence of multiple pairs of relationships between properties or relationships between figures. While the focus of the response is on more than one relationship, they are treated in isolation. Minimisations remain in conflict with the need to distinguish a certain figure from other figures within the same global class.	There is an unprompted acceptance of a class of figures containing subsets. While this notion of class inclusion is accepted and utilised, it is not justified adequately.
R_1(F) response	The response includes a focus upon the network of relationships among known property and figure relationships. The interrelationships may not incorporate all property relationships.	The notion of class inclusion is an integrating feature of the response. A class of figures incorporates subsets, which are inclusive of generic categories identified by other names. Each class maintains a workable identity while the focus is upon the network of relationships.
U_2(F) response	A network of relationships is the focus of the response. There is an understanding of the general overview, which utilises relationships among groups of properties and figures. The notion of minimisation can be held in more than one circumstance spontaneously.	The notion of class inclusion acquires further development. Conditions are placed upon the classes of figures, which acknowledge more than one system of relationships. This requires an overview of the interrelationships.

The study reported here was designed to provide a quantitative synthesis of the developmental frameworks that described students' understandings of the relationships among figures and properties. The research questions addressed are the following.

1. How do the identified response categories reflect the hierarchical framework of the SOLO model?

2. Is there an order of difficulty among the item responses, which can assist in interpreting the complexity of students' responses to tasks concerning relationships among figures and relationships among properties?

3. Which response categories to tasks had relatively larger increases in complexity from the prior response category, concerning students' understandings of relationships among figures, and relationships among properties?

Methodology

The previous qualitative study involved in-depth interviews with 24 students of higher mathematical ability, purposely selected, within Years 8–12 (ages 13–18 years) in two secondary schools. There were equal numbers of males and females. Twelve of these students repeated the interview tasks two years later, and hence the data set to be analysed comprises a total of 36 sets of student responses.

The nature of the qualitative study was to have the students complete seven tasks that focused upon known relationships among figures and among properties within the contexts of triangles and quadrilaterals. Seven items were included in the interview protocol. The tasks provided a catalyst for discussion that enabled prompts and probes as appropriate. The duration of each interview was approximately 1 hour. Further details of the interview are presented in Serow (2006) and Currie and Pegg (1998). An outline of interview tasks (items) is contained in Table 2 below.

Table 2.
Item Focus and Descriptors

Item	Focus of the Item	Item Descriptors
1	Relationships among triangle figures	Design a tree diagram that links the different triangles (equilateral, right isosceles, acute isosceles, obtuse isosceles, right scalene, acute scalene, and obtuse scalene). Discussion follows concerning the reasons for links and/or lack of links.
2	Relationships among quadrilateral figures	Design a tree diagram that links the different quadrilaterals (trapezium, square, rectangle, rhombus, parallelogram, kite). Discussion follows concerning the reasons for links and/or lack of links.
3	Relationships among equilateral triangle properties	After selection of known property cards for the equilateral triangle, the student is asked to provide a minimum combination of cards to enable a friend to identify the shape with accuracy. Multiple combinations were then requested.
4	Relationships among right isosceles triangle properties.	Task above repeated for the right isosceles triangle.
5	Relationships among square properties.	Task above repeated for the square.
6	Relationships among parallelogram properties.	Task above repeated for the parallelogram.
7	Relationships among rhombus properties.	Task above repeated for the rhombus.

Each of the responses to the seven tasks was coded according to the SOLO codings described in Table 1. The results presented in this paper are a review of the Rasch results across the seven items and 36 student response sets. With the categories of each item being of an ordinal nature, the data assumptions of the QUEST application of the Rasch modelling process are consistent with the data of this study. The partial credit model was used to provide data concerning the relatively larger distances between response categories and clusters of response categories. The data set is combined to allow a conservative comparison of the item response categories on a single hierarchical line of inquiry (Bond & Fox, 2001).

Results

Reliability

Item separation reliability statistics produced by the QUEST software are described by Adams and Khoo (1993) as the proportion of the observed variance that is considered true. In this study, the relatively small sample size across a limited number of grades meant that the item separation reliability was low, due to larger measurement error. Due to this factor, the item estimates are to be interpreted conservatively and the results are presented in clusters of response categories. Even though the item separation reliability was low, there are some points of interest in terms of the relative difficulties among the response categories and this is the focus of the paper.

Fit Statistics

Fit statistics are the means and standard deviations of the infit (weighted) and outfit (unweighted) fit statistics in the mean square form. When the observed data and estimates are compatible, the expected value of the infit mean square is close to 1 (1.02) with a small standard deviation (0.17), and the transformed infit (Infit t) is close to zero (0.12). Hence, the items come from the same underlying construct, namely, relationships among figures and relationships among properties.

The component infit mean square values are presented in graphical form in Figure 1 to assist in interpretation. The infit statistic for each item is the weighted residual based statistic, which indicates quantitatively how appropriately each item fits the model (Fisher, 1993). This comparison can be used to confirm the unidimensionality of the items, confirming construct validity of the items. Fit is acceptable if the mean lies between 0.77 ad 1.3 (Keeves & Alagumalai, 1999), in this case the infit mean is 1.02.

The figures on the horizontal scale represent the infit mean square scale and the asterisks indicate the magnitude of the fit statistic for each item on the same line. Fit statistics that lie within the two dotted vertical lines are considered acceptable. The well-fitting nature of the items to the model indicates that the items represent aspects of a latent trait. The infit mean square map for the seven items, which appears below in Figure 1, indicates that six of the seven items are within the acceptable limits. Item 4, which concerns students' understanding of the relationships among properties of the right isosceles triangle, is only slightly to the right-hand side of the acceptable limits. This indicates that for Item 4, there is an element of randomness in coding.

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INFIT MNSQ
           .63     .71       .83       1.00      1.20      1.40
-----------+--------+---------+---------+---------+---------+-----
  1 item 1              .            *|              .
  2 item 2              .        *    |              .
  3 item 3              .   *         |              .
  4 item 4              .             |              .*
  5 item 5              .             *              .
  6 item 6              .         *   |              .
  7 item 7              .             |    *         .
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Figure 1. Item map.

Item Difficulty

The information pertinent to item estimates is displayed in the variable map in Figure 2. There are seven tasks in total, and 36 sets of student responses represented. The chart includes a logit scale on the left of the diagram on which both items (*n*=7) and cases (*n*=36) are calibrated. The distribution of students is represented by XXXs on the left-hand side of the chart. The seven tasks are identified in Figure 2 as:

1. Relationships among triangle figures.
2. Relationships among quadrilateral figures.
3. Relationships among equilateral triangle properties.
4. Relationships among right isosceles triangle properties.
5. Relationships among square properties.
6. Relationships among parallelogram properties.
7. Relationships among rhombus properties.

LOGIT	CASES	ITEMS			
3					
	X				
				$5.U_2$(F)	
2					
				$5.R_1$(F)	
		$1.U_2$(F)			
				$6.U_2$(F)	$7.U_2$(F)
		$2.U_2$(F)			
1	X			$3.U_2$(F)	
		$1.R_1$(F)		$3.R_1$(F)	$4.U_2$(F)
	XXX	$1.M_1$(F)		$6.R_1$(F)	
		$1.U_1$(F)	$2.R_1$(F)	$4.M_1$(F)	
	X			$3.M_1$(F)	
	XXX			$7.M_1$(F)	
	XXX				
	XX	$2.R_2$(CS)			
	XX			$5.M_1$(F)	$6.M_1$(F)
0	XX	$1.R_2$(CS)			
	XXX				
	X			$3.U_1$(F)	$4.U_1$(F)
	XX			$6.U_1$(F)	$7.U_1$(F)
	XXXX			$3.R_2$(CS)	$5.U_1$(F)
	XXXX			$4.R_2$(CS)	
	XX				
				$7.R_2$(CS)	
				$6.R_2$(CS)	
		$2.M_2$(CS)			
-1	X			$3.M_2$(CS)	$5.R_2$(CS)
		$1.M_2$(CS)			
	X	$2.U_2$(CS)		$6.M_2$(CS)	
				$4.M_2$(CS)	$5.M_2$(CS)
				$6.U_2$(CS)	$7.M_2$(CS)
-2		1.U2(CS)			

Figure 2. Item and case estimates (thresholds).

Item Analysis

The following discussion addresses the patterns that have emerged concerning item difficulty across item response categories. A comparison of item difficulties across items concerning figures and property relationships follows. The comparison involves individual response category item difficulties, which appear in Table 3.

Table 3
Item Response Category Difficulty Levels

	Concrete Symbolic			Formal			
FIGURES	U_2	M_2	R_2	U_1	M_1	R_1	U_2
Triangles	-2.06	-1.10	0.07	0.74	0.82	0.94	1.42
Quadrilaterals	-1.28	-0.97	0.22			0.70	1.16
PROPERTIES							
Equilateral		-1.08	-0.35	-0.20	0.58	0.91	1.08
Right Isosceles		-1.44	-0.50	-0.12	0.72		0.88
Square		-1.44	-1.03	-0.36	0.19	1.72	2.09
Parallelogram	-1.63	-1.26	-0.77	-0.32	0.16	0.81	1.23
Rhombus		-1.63	-0.74	-0.28	0.48		1.24

Similarities and differences in relation to degree of difficulty and characteristics of the responses form the basis of the comparison. This is considered in clusters of item responses beginning with the lower level SOLO responses, which also appear at the lower end of the item estimate threshold.

In the tasks concerning relationships among figures, and those concerning relationships among properties, a hierarchical framework emerged that is evident in the SOLO categorisations and is reinforced by the application of the Rasch analysis. Each of the items followed the SOLO sequence of levels within cycles without exception. The following discussion provides a comparison of item estimate thresholds when comparing item difficulty across clusters of response categories concerning relationships among figures, and item responses concerning relationships among properties.

The U_2(CS) response category concerning relationships among triangle figures was found by the sample of students to be of the lowest degree of difficulty. This was followed by other groups of U_2(CS) and M_2(CS) responses concerning relationships among figures, and relationships among properties. Hence, the students found the utilisation of the three mutually exclusive classes of triangles at a similar degree of difficulty to focusing upon unique property signifiers of figures with reference to the figure only. It appears that the progression to finding multiple properties that are unique to a figure assists in the formation of minimum combinations to encapsulate multiple properties to form generic categories. Although restrictive language, which does not facilitate the inclusive nature of properties, is utilised in U_2(CS) responses concerning figures and properties, this level is a necessary precursor for developing notions of minimum property combinations.

Next on the logit scale is a cluster of R_2(CS) responses including all five tasks concerning relationships among properties. Hence, the students found ordering between two properties to be at a similar degree of difficulty in both the triangle and quadrilateral contexts. Although the U_1(F) responses are grouped together when addressing tasks concerning the relationships among properties, these appear before the R_2(CS) responses in the context of relationships among figures, thus indicating that the students found a focus upon relationships between pairs of properties and/or figures, and making property links across classes, of a similar degree of difficulty in both triangles and quadrilaterals contexts. The U_1(F) response concerning property relationships appears to be a precursor to focusing upon relationships among classes of figures, which are not supported by visual cues. The remaining first cycle formal responses are clustered at a similar degree of difficulty, thus indicating that the utilisation of a single network of relationships among figures, utilising multiple relationships among properties, and an attempt to focus upon the interrelationships among property relationships are at a similar degree of difficulty.

The U_2(F) responses have a greater range in terms of degree of difficulty. This final cluster indicates that the students found the focus upon more than one network of relationships involving notions of class inclusion, and the focus upon the network of relationships to form minimisations, the most difficult groups of responses. In the context of property relationships the students found the right isosceles triangle and parallelogram items the least difficult at this SOLO level. Class inclusion notions requiring the acknowledgment of multiple subsets when relating figures were at a similar degree of difficulty to the utilisation of the network of relationships among properties of the equilateral triangle. This was closely followed by the rhombus task.

It is interesting to note the high degree of difficulty found by the sample of students when forming minimisations of square properties based upon the network of property relationships. Although this indicates that the lower SOLO categories indicated a comparatively lower degree of difficulty for the square item compared with other items of the same SOLO level, the shift required to move from M_1(F) to R_1(F) is relatively difficult in the context of the square. The responses indicated that this is due to factors such as visual cues assisting links, and multiple unique properties of the square that assist understanding at lower SOLO levels. In contrast, at the formal mode the student must leave the real world referent behind and focus upon the network of relationships among the properties, as opposed to concrete symbolic justifications.

The degree of difficulties between item response categories, known as step difficulties, further clarifies the similarities and differences among the SOLO categorisations. The step difficulties describe the change in degree of difficulty, found by the sample of students, between one SOLO level and the subsequent SOLO level. These appear in Table 4, and also include the mean step difficulty for each SOLO response category.

Table 4
Step Difficulties

	Concrete Symbolic		Formal			
	U_2 to M_2	M_2 to R_2	R_2 to U_1	U_1 to M_1	M_1 to R_1	R_1 to U_2
Item 1	0.96	1.17	0.67	0.08	0.12	0.48
Item 2	0.31	1.19				0.46
FIGURES MEAN	0.64	1.18	0.67	0.08	0.12	0.47
Item 3		0.73	0.15	0.78	0.33	0.17
Item 4		0.94	0.38	0.84	0.16	0.25
Item 5		0.41	0.67	0.55	1.53	0.37
Item 6	0.37	0.49	0.45	0.48	0.65	0.42
Item 7		0.89	0.46	0.76		
PROPERTIES MEAN	0.37	0.70	0.42	0.68	0.67	0.30

Of particular interest, are the higher and lower step difficulties. The step difficulty between a U_2(CS) response and an M_2(CS) response concerning relationships among figures has a mean of 0.64. It was also found to be difficult by the sample of students to respond at R_2(CS) compared with M_2(CS) concerning relationships among figures (mean 1.18). This was similar to the step difficulties concerning relationships among properties, where M_2(CS) to R_2(CS) (0.70) was found to have a comparatively high step difficulty.

In addition, movement through the first cycle of the formal mode is a difficult progression concerning relationships among properties. This is evident by: U_1(F) to M_1(F) (mean 0.68) and M_1(F) to R_1(F) (mean 0.67). It is interesting to note that the highest individual step difficulty concerns the shift from M_1(F) to R_1(F) in regards to relationships among square properties (1.53). Overall the progression from U_1(F) to M_1(F) concerning relationships among figures has the least step difficulty (0.08).

Discussion

The study was designed to complement and extend a qualitative analysis of results, through a procedure that provided comparative qualitative results across relationships among figures, relationships among properties, and different contexts. Of particular interest was the finding that despite the quadrilateral context being chosen in the study due to an increase in complexity, this was not mirrored by the analysis. The degree of difficulty was found to be similar within the triangle and quadrilateral contexts. The application of the Rasch model supported the developmental sequence that evolved through the SOLO categorisations. The results also highlighted a number of interesting trends. The first of these is the consistency of the groupings evident in the item estimate thresholds when comparing student responses across figure tasks, property tasks, and different contexts. Secondly, the fit statistics and item estimates indicate that the items came from the same underlying construct. This provides confirmation of the appropriateness of the SOLO model.

The concrete symbolic responses indicate that a focus upon a single property to encapsulate separate classes of figures is a prerequisite to focusing upon a single unique property of a figure when asked to provide a minimum description of a figure. The M_2(CS) responses indicate that the shift in moving from multiple properties to form individual classes of figures is at the same level as identifying multiple unique property signifiers while maintaining a real world referent. Thus, the figure determines the properties.

The identification of a link between two properties, and the shift to utilising the relationship as a workable unit, are necessary precursors to the utilisation of relationships among classes of figures without the need for a real world referent. This progression is a shift into the formal mode in terms of relationships among properties, and is characterised by the property relationships determining the figure in both contexts. When the formal mode is entered, concerning relationships among properties, the degree of difficulty is the same in regards to linking properties or figures despite the bifurcation. The focus upon perceiving the property relationships as determining the figures and utilising inclusive language to describe properties begins at a lower level than focusing upon links across classes of figures. This sequence flows through to a focus upon the network of relationships among figures and properties where there is greater variation in degree of difficulty found by the students across the seven tasks when providing a U_2(F) response.

The higher and lower step difficulties between SOLO response categories assist in the interpretation of the more difficult, and less difficult progressions from one SOLO level to the subsequent SOLO level. The highest increases, or “hard boundaries”, were found to be in the second cycle of the concrete symbolic mode concerning relationships among figures. These increases concerned the progression from a focus upon single properties to form individual classes of figures, to multiple properties while maintaining mutually exclusive classes, with the hardest boundary being the shift to a focus upon relationships between classes that are not supported by dominant visual differences. Similarly, in the context of property relationships, hard boundaries exist in the shift from a focus upon multiple properties as unique signifiers of a figure, to a focus upon an ordering between two properties. To a lesser extent, the boundary is relatively difficult when moving from an understanding that the figure determines the property, to a shift into the formal mode where relationships among properties determine the figure. Another boundary exists in the progression from a focus upon multiple relationships among properties, to an overview of the network of relationships among properties.

Of particular interest are the supporting influences between relationships among figures, and relationships among properties. These include the encapsulation of properties to form classes, a shift to perceiving the properties as determining the figure, the dominance of recognised similarities and differences across classes of figures and among properties, and the utilisation of inclusive or exclusive, class, or property descriptions. The identification of differing boundaries between the categories provides insight into the difficulties found by students when encountering notions of class inclusion in Geometry.

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