Students’ Emerging Algebraic Thinking in the Middle School Years

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There is a strong case for arguing that the application of relational thinking to solve number sentences embodies features of mathematical thinking that are centrally important to algebra. This study investigates how well students in Years 5, 6, and 7 in three countries were able to use relational thinking to solve different types of number sentences. There were other students who appeared to rely solely on computational method to solve the same number sentences. The study then examined whether those who had shown clear evidence of relational strategies to solve the number sentences were better placed to solve symbolic sentences than those who had used only computational methods on these number sentences.

Relational Thinking

In their study, “The algebraic nature of students’ numerical manipulation in the New Zealand Numeracy Project”, Irwin and Britt (2005) argue that the methods of compensating and equivalence that some students use in solving number sentences may provide a foundation for algebraic thinking (p. 169). These authors give as an example the number sentence 47 + 25 which can be transformed into 50 + 22 by “adding 3” to 47 and “subtracting 3” from 25. They claim (p. 171) “that when students apply this strategy to sensibly solve different numerical problems they disclose an understanding of the relationships of the numbers involved. They show, without recourse to literal symbols, that the strategy is generalisable.” Several authors, including Stephens (2006) and Carpenter and Franke (2001), refer to the thinking underpinning this kind of strategy as relational thinking.

Solving number sentences successfully using relational thinking certainly calls on a deep understanding of equivalence. Students need to know the direction in which compensation has to be carried out in order to maintain equivalence (Kieran, 1981; Irwin & Britt, 2005; Stephens, 2006). Some children who correctly transform number sentences involving addition reason incorrectly that a number sentence such as 87 – 48 can be transformed to be equivalent to 90 – 45. These children do not understand the direction in which compensation must take place when using subtraction or difference. They fail to recognise that the relationship of difference is fundamentally different from addition. Other children, however, recognise this feature explaining that in order for the difference to remain the same, the same number has to be added to (or subtracted from) each number to the left of the equal sign. These children write correctly 87 – 48 = 89 – 50. The first part of this study probed children’s thinking with number sentences.

The Study

Three groups of number tasks shown in Figure 1 were given to students in Years 5, 6, and 7 using a pencil-and-paper questionnaire administered in regular class time. In introducing the questionnaire, classroom teachers told students that:

> This is not a test. It is a questionnaire prepared by researchers … looking at how students read interpret and understand number sentences. For most of the questions there is more than one way of

> giving a correct answer. Please write your thinking as clearly as you can in the space provided after each question and don't feel that you have to write a lot.

The questionnaire and the teacher's introduction were translated into Japanese and Thai. Each group of problems, shown in Figure 1, was introduced with the words: "Write a number in each of the boxes to make a true statement. Explain your working".

Group A (on one page)	Group B (on one page)	Group C (on two pages)
23 + 15 = 26 + □	39 – 15 = 41 – □	746 – 262 + □ = 747
73 + 49 = 72 + □	99 – □ = 90 – 59	746 + □ – 262 = 747
43 + □ = 48 + 76	104 – 45 = □ – 46	
□ + 17 = 15 + 24		

Figure 1. Three groups of missing number sentences.

The study involved three cohorts of students Japan (277 students), Australia (301 students) and Thailand (194 students). Two schools were used in each country with students in Years 5, 6, and 7 approximately the same age (10 years old to 13 years old). In all schools involved in the study the teaching of computational algorithms forms a key part of the curriculum. Even if relational approaches are taught in some schools, they are not given the same time or emphasis as computational approaches. In Australia and Thailand, the study was carried across all year levels at the one time. In the case of Japan, Year 5 was tested at the end of one school year and Year 6 and Year 7 at the start of the next school Year. For this reason, the Japan results for Year 5 and Year 6 are considered together, whereas Year level results for Thailand and Australia are separated.

Evidence of Relational Thinking

Relational thinking is evident when, for example, verbal descriptions, arrows, or diagrams are used to compare the size of numbers either side of the equal sign, and where these verbal descriptions, arrows or diagrams are used in chain of argument, based on uncalculated pairs, using compensation and equivalence to find the value of a missing number. By contrast, computational thinking follows a fixed pattern. These features were discussed more fully in Stephens (2004, 2006).

In Group A and B questions, students must complete the calculation on the opposite side to where the □ is shown, and use this result to find the value of the missing number. For example, in the first problem of Group B, students must first find 39 – 15; and having found this to be 24, they then need to find the number which taken from 41 gives a result of 24 (or which added to 24 gives 41) for which the result is 17. In Group C, students must first subtract 262 from 746 giving 484, before proceeding to find the missing number by subtracting 484 from 747.

For each group of questions a benchmark sample was prepared, illustrating each score. Each student's work was checked independently by two markers. A high degree of consistency was evident across markers in all three countries. Whenever there was disagreement between markers, this was usually resolved by the markers themselves – usually one had missed an important clue. Very rarely, such disagreements were referred to a supervising researcher. Two student responses showing very clear relational thinking are given for each group of items in Figure 2.

Group A

- If I take 2 from 17 and add 2 to 22, it is the same as the number sentence after it. (Year 6 student)
- In 43 + □ = 48 + 76, 43 to 48 is + 5, 81 to 76 is – 5. These are equivalent, as you've done the same action to both sides. (Year 7 student)

Group B:

- As 99 is 9 more than 90, the missing number must be 9 more than 59. Therefore the answer is 68. (Year 5 student)
- I added 1 to 104 and 45. As long as I add the same number to both, it (104 – 45) will stay equivalent. (Year 6 student)

Group C:

- 746 is one less than 747, so 262 is one less than the answer. My answer is 263. (Year 5 student)
- 746 is 1 unit less than 747, so if you add 263 you will only need to minus 1 unit less than 263 for the equation to be equal on both sides. (Year 7 student)

Figure 2. Selected students' responses showing relational thinking.

Scoring procedures. Each group of problems was scored using a five-point scale shown in Figure 3. Thus, a single score was assigned to each group of questions even if children did not solve each question in the same way. This scoring scheme which had been validated for an earlier study (Stephens, 2004) was applied to Groups A, B, and C.

0 – arithmetical thinking evident for all questions; for example, through evidence of progressive calculations and use of algorithms to obtain results for additions and subtractions, even where these approaches resulted in incorrect answers, and no evidence of any relational thinking; also where an answer only has been given with no working shown to indicate what method has been used

1– a clear attempt to use relational thinking in at least one question, but not successfully executed (e.g., in Group B by giving answers of 13, 50 and 103)

2 – relational thinking clearly and successfully executed in one question, even if other problems are solved computationally or by incorrect relational thinking

3 – relational thinking clearly and successfully executed in at least two questions, but where the remaining question or questions are not solved relationally or solved using incorrect relational thinking

4 – all questions are solved clearly and successfully using relational thinking, even if computational solutions are also provided in parallel.

Figure 3. Scoring rubric.

Results of the Questionnaire

Clear evidence of relational thinking was present across all three Groups of questions among Japanese and Australian students. In the Japanese Year 5 and 6 cohort, almost 40% of students achieved a Score 4 (accomplished relational thinking) on Group A. The proportion of Score 4 was nearly 25% for Group B, and a little less than 20% for Group C. On the other hand, the proportion of Year 5 and 6 students who obtained Score 0, by using clear computational approaches or providing no evidence of relational thinking, ranged from about 35% for Group A, to 40% for Group B, and 65% for Group C. By Year 7, the proportion of Score 4 performances increased for all three groups of questions. This increase was not offset by an equivalent fall in the proportion of Score 0 performances that fell only slightly from Years 5 and 6 to Year 7. The increase in Score 4 performances in the Year 7 cohort was matched by reductions in the proportion of Scores 2 and 3. Although the Japanese mathematics curriculum seems to favour the development of relational approaches among many students, many other students still seem unable to or prefer not to use them.

The vast majority of Thai students used computational approaches in all three Groups of questions. In Year 5, no Thai student scored 4 on any group of questions. In Year 5, the proportions of Score 0 were 83% (Group A), 92% (Group B) and 98% (Group C). In Year 6 the average of Score 0 across the three groups of questions was 90%. In Year 7, it was 84% with gradual increases in the proportion of students in Years 6 and 7 achieving between Score 2 and Score 4. The gradual emergence of relational thinking in the Thai cohort seems more likely to be the result of individual student insight rather than an intended result of the mathematics curriculum.

The two Australian schools showed wide variation in the use of relational strategies. Looking only at the Year 6 cohorts in the two schools, the proportion of Score 0 results for Group A, B, and C questions in School 1 was 60%, 64%, and 78% respectively, compared to 34%, 32%, and 48% in School 2. Similarly, the proportion of Score 4 results for Group A, B, and C questions in School 1 was 25%, 9%, and 16% respectively, compared to 48%, 30%, and 46% in School 2. The reason for this marked difference is that in School 2 relational approaches are featured explicitly in the mathematics curriculum, whereas in School 1 they seem not to be emphasised.

Stability of Thinking

How consistent were students in their use of relational or computational approaches across the three groups of problems? Students were classified into three groups: those students who used relational strategies across all three groups of problems (SR–Stable Relational); those students who used only arithmetical or computational approaches across all three groups of problems (SA–Stable Arithmetical); and those students whose thinking was not consistent across the three groups (NS–Not Stable). The following rule was used.

SR: if student scored ≥ 1 on each of Group A, B, and C
SA: if student scored 0 on each of Group A, B, and C
NS: if student scored ≥ 1 on one or two of Group A, B, or C; and 0 on other(s).

A criterion of ≥ 1, instead of ≥ 2, across the three groups as evidence of stable relational thinking was justified because a score of 1 on Group B was without exception associated with successful relational thinking (≥ 2) for Group A and/or Group C questions. Aside from responses to Group B where students compensated in the wrong direction, a score of 1, indicating incorrect relational thinking, was very rarely given for responses to Group A and Group C questions.

How do Relational Thinkers Deal Successfully with Symbolic Sentences?

What evidence is there that students who successfully apply relational thinking to solve number sentences are able to extend these processes to solve sentences that are explicitly algebraic? Linchevski and Livneh (1999) point to the structural relations that students need to understand from arithmetic if they are to move successfully into algebra. MacGregor and Stacey (1999) also contend that deeper understanding of numerical operations is linked to later success in algebra. Using symbolic terms makes it more difficult for students to use computational checks. Some students solve symbolic expressions, such as $x + 3 = 21$, by drawing on their knowledge of number facts, or using guess-and-check methods. But another type of symbolic sentence, true for all values of the literal symbol, can be used to probe students' understanding of the meaning of symbolic expressions. This type of question, shown in Figure 4, was used to probe whether there is a clear link between successful application of relational thinking applied to number sentences and students' ability to understand the structure of symbolic sentences.

Place the four numbers n – 1, n + 5, 7 and 1 in the four boxes below so that the statement is always true.
☐ + ☐ = ☐ + ☐
Box A Box B Box C Box D
Explain why your answer is correct.

Figure 4. Making a sentence that is always true.

Students in Years 5, 6, and 7 in three countries had not been introduced to "always true" symbolic expressions. (Of course, many students by Year 5 have met single-value missing number sentences, such as ☐ + 3 = 21.) Some students did not attempt the question, or they wrote a sentence which is not true for all values of n, for example by writing a sentence which has the four numbers in boxes in the order in which they appear in the question. Some other students wrote a correct sentence but could not explain why it was true for all values of n. On the other hand, several possible approaches were used by students to explain why their sentence is always true. These various possibilities informed the partial-credit scoring rubric shown in Figure 5 used to grade students' responses.

NR – no response to the question involving literal symbols and number terms
Score 0 – incorrect or inadequate relation, no evidence of relational thinking
Score 1 – correct relation shown but no explanation given
Score 2 – correct relation shown, and correctly illustrated with one or more numerical values
Score 3 – correct relationship shown, and successfully illustrated by showing a balance with respect to the numbers, "ignoring" n terms; or by generally referring to balance among terms
Score 4 – correct relationship shown, and explained by explicit reference to the numbers and the n terms being equivalent on both sides, whatever the value of n, or by showing that the same algebraic structure exists on both sides.

Figure 5. Rubric used for scoring question involving literal symbols.

The following responses formed a benchmark sample for a score of 4, 3, or 2 for this question.

Exemplifying Score 4. A Year 6 student, having written, $7 + n - 1 = 1 + n + 5$, said:

> "This answer is correct because you will always get an answer 6 more than n, because n less 1 plus 7 will give us 6 more than n. Also because n more than 5 plus 1 will give 6 more than n. This will have a lot of different answers but you will always get an answer 6 more than n."

A Year 7 student wrote $n - 1 + 7 = n + 5 + 1$, and explained:

> "My answer is correct as no matter what n is, $n - 1$ is 6 units less than $n + 5$. This is balanced as 7 is six units more than 1."

Exemplifying Score 3. A Year 7 student wrote $n - 1 + 7 = n + 5 + 1$, and wrote:

> "7 and $n - 1$ become 6; $n + 5$ and 1 become 6. Both sides are equivalent to 6".

Exemplifying Score 2. A Year 5 student wrote $1 + n + 5 = 7 + n - 1$, and then let $n = 5$ showing that

> $1 + 5 + 5 = 7 + 5 - 1$. No reason was offered to show why the statement is always true.

There were some clear associations between highly accomplished explanations (Score 4) given to this question involving literal symbols and accomplished relational thinking used on the number sentences. For example, in Japan in Years 5 and 6, all 6 students who scored 4 on the question involving literal symbols also scored at least one 4 on the number questions. In Japan, where 54 Year 7 students scored 4 on this question, 44 showed very clear relational thinking on the number sentences, even if this was not always scored as high as a 4. In Australian School 2, the same applied to all 10 students in Years 5 and 6 who scored 4 on this question. Further, no student in Years 5 and 6 in any of the three countries who scored 0 on all three groups of number sentences scored 4 on the question involving literal symbols. This pattern was almost perfectly replicated in Year 7 cohorts.

Very many students who gave highly accomplished responses (Score 4) to this question applied compensation to the two terms involving literal symbols and to the two number terms, showing equivalence, whatever the value of *n*. Is there a clear connection between relational thinking on number sentences and success on the question involving literal symbols? Put most simply, one might expect a strong connection between those students who were classified as Stable Relational (SR) thinkers on the three groups of number sentences and their success in dealing with the question using literal symbols. A consequence of this "strong" position, if it were true, is that students who were classified as Stable Arithmetical (SA) on the three groups of number sentences would be less likely to deal successfully with the question involving literal symbols. These positions are now analysed.

Using Relational Thinking on Number Sentences (SR) as a Predictor

The following table gives the numbers of students who were classified as SR who also obtained a score of ≥ 1 on the question involving literal symbols (SR/LS). Their success rate is then compared to the percentage of their cohort in dealing successfully (i.e., obtained a score of ≥ 1) with the question involving literal symbols (LS).

Table 1
Using Stable Relational Thinking (SR) as a Predictor

Country	Cohort		Number of SR students	Number (%) SR/LS		Number (%) of LS in cohort	
Japan	Year 5/6	N = 133	41	32	78%	70	53%
	Year 7	N = 144	56	55	98%	127	88%
Australia (School 1)	Year 5	N = 41	8	4	50%	13	32%
	Year 6	N = 45	8	0	0%	3	7%
	Year 7	N = 44	9	7	77%	27	61%
Australia (School 2)	Year 5	N = 50	13	7	54%	17	34%
	Year 6	N = 50	27	22	81%	31	62%
	Year 7	N = 71	49	44	92%	58	82%
Thailand	Year 5	N = 53	N/A	N/A		N/A	
	Year 6	N = 64	N/A	N/A		N/A	
	Year 7	N = 77	4	2	50%	21	31%

This criterion seems to work well in Years 5 and 6 in Japan and Australian School 2 where the number of students classified as SR is comparatively high. In these two groups, the success rate of students who showed stable relational (SR) performance on the three groups of number sentences was almost 20% higher in obtaining a score ≥ 1 on the question involving literal symbols than the general success rate. The strength of connection is not as strong in both groups in Year 7 where the success rate of the SR performers on the number sentences is only 10% higher than the general success rate. Ceiling effects begin to emerge in the Year 7 in Japan and in Australian School 2 where 88% and 82% respectively of students in Year 7 were able to deal successfully (Score ≥ 1) with the question involving literal symbols.

However, serious difficulties exist in the application of the criterion in Year 5 and Year 6 the Australian School 1 and in the Thai cohort where few students were able to be classified as SR on the number questions, and where few were also successful on the question involving literal symbols. The criterion could not reasonably be applied in the case of Years 5 and Year 6 in the Thai cohort where only one student was classified as SR; and where in Year 5 only two students scored ≥ 1 on the question involving literal symbols. In Thailand in Year 6, however, 12 students scored ≥ 1 on the question involving literal symbols, despite the paucity of stable relational (SR) thinkers on the number sentences. Even in the Year 7 Thai cohort, the number of students classified SR was too small (4) to allow any reliable predictions. Similar difficulties also occur in Australian School 1 where only 3 students in the entire Year 6 sample scored ≥ 1 on the question involving literal symbols.

Using Arithmetic Thinking (SA) as a Predictor

How well did those students who met the criterion for Stable Arithmetic (SA) – that is, those who scored 0, 0, 0 on all three Groups of number sentences – perform on the question involving literal symbols? Given the difficulties applying the preceding test to the entire Thailand cohort and to Australian School 1, this test becomes more important. In Australian School 1 in Year 6, 23 students scored 0 on all three groups of number sentences. Of these 23, 21 were graded either NR or 0 on the question involving literal symbols, with only one of the 23 obtaining a 1 for this question, and one other obtaining a 2. In Year 5, 28 students got a 0 on all three groups of number sentences. Of these 21 got either NR or 0 (no success) on the question involving literal symbols, with four obtaining 1 for this question, and three obtaining a 2.

Likewise, in the Thailand cohort, there is a strong connection at each Year level between SA thinking on number sentences and failure to deal successfully with the question involving literal symbols. However, even for this cohort, the strength of this connection declines with each additional Year level. With each successive year level, more students classified as SA on the number sentences are able to score ≥ 1 on the question involving literal symbols. These results across all cohorts of Years 5, 6, and 7 students are given in Table 2.

The predictive value of this criterion seems to be strongest in Years 5 and 6 in all three country samples. Its predictive force is still quite strong in Thailand in Year 7; much less so in Year 7 in the Australian schools; and not at all in Year 7 in Japan. It may be argued that by Year 7 more students are familiar with literal symbols and so are able to deal successfully with the question involving literal symbols.

Table 2
Using Stable Arithmetical Thinking (SA) as a Predictor

Country	Cohort		Number of SA students.	SA students with no success on literal symbol question	
Japan	Year 5/6	N = 133	37	25	68%
	Year 7	N = 144	43	12	28%
Australia (School 1)	Year 5	N = 41	28	21	75%
	Year 6	N = 45	23	21	93%
	Year 7	N = 44	24	13	54%
Australia (School 2)	Year 5	N = 50	18	16	89%
	Year 6	N = 50	15	9	60%
	Year 7	N = 71	11	5	45%
Thailand	Year 5	N = 53	43	41	95%
	Year 6	N = 64	51	44	86%
	Year 7	N = 77	55	44	80%

There are some students, more in Japan and Australia than in Thailand, who are able to adopt relational thinking for the question involving literal symbols, even though they showed no evidence of relational thinking on the number sentences. These students can exercise choice; they are able to apply relational strategies when required in the case of the sentence involving literal symbols. For example, in the Japanese Year 5 and 6 cohort, 37 students obtained 0 on all three groups of number sentences, with 25 of these receiving either NR or 0 for the question involving literal symbols. Of the remaining 12 students, five received a score of 1, four a score of 2, and three a score of 3. The competent performances (Score 2 and Score 3) of these 5 students had not been preceded by any relational thinking in their work on number sentences. In Australian School 2 in Year 6, of the 15 students who scored 0 on all three groups of number sentences, 9 of these received either NR or 0, but three students received a score of 1 on the question involving literal symbols, and a further three also obtained a score of 2. By Year 7 in Thailand, 11 students classified as SA (0, 0, 0) on the number sentences achieved scores ranging from 1 to 3 on the question involving literal symbols.

Discussion of Limitations and Future Directions

In statistical analyses where some clear associations are present but not definitive, it is important to ask why this is so. The first and most obvious comment is that the three groups of number sentences may not have separated those who were capable only of thinking computationally from those who chose to solve the number sentences computationally but who could have used relational approaches to solve these sentences if pressed to do so. Some of these "computational" students applied relational approaches to deal more or less successfully with the expression involving literal symbols. Students who are competent calculators may prefer that approach even though it is much more demanding than relational thinking in the case of Group C, and somewhat more demanding in the case of Group A and B questions.

It should be remembered that no student who consistently solved the number sentences computationally was able to achieve the highest score (Score 4) on the question involving literal symbols, although there were quite a few who produced an expression in the correct form but with no explanation (Score 1) and others who were able to justify their choice of a correct literal expression by using one or more values of the literal symbol (Score 2). Those with Score 1 who produced an expression in the correct form – without explanation or justification – may have used strategies such as "guess-and-check" that fall a long way short of deep relational thinking.

It is also clear that some students who appeared to be stable relational thinkers (SR) did not deal successfully with the question involving literal symbols. Among this latter group might be those who solved only some of each group of number sentences relationally. It is a big jump from being able to apply relational thinking to complete an already formed number sentence to being able to construct and justify an "always true" sentence involving literal symbols and numbers in an equivalence relation. Although the findings of this study support the view of Linchevski and Livneh (1999) that many of algebraic relations met by students inherit the structural properties associated with number sentences with which students are, or should be, familiar, it is clear that the missing number questions were not sufficiently sensitive to elicit and identify the kind of relational thinking that students needed in order to solve the question involving literal symbols.

Some students may have used grouping and simplification techniques to deal with the question involving literal symbols even if they had chosen to solve by computation all the number sentences. From our study of the curriculum documents of the three countries we were confident that students in Year 7 had not been taught these techniques, but this cannot be ruled out for every student.

Is it possible to introduce an extra question that would press those who chose to solve the number sentences computationally to disclose any latent relational understanding, and at the same time to discriminate among relational thinkers? To these purposes, a question modelled after the research programme, Concepts in Secondary Mathematics and Science, (CSMS, see Hart, 1981) might ask students:

What can you say about c and d in the following mathematical sentence?

$$c + 2 = d + 10$$

If equivalence and compensation are at the heart of relational thinking, the goal of this question is to have students say that this sentence will be true for *any* values of c and d provided c is 8 more than d. But there are intermediate responses that fall short of this understanding. Computational thinkers are likely to be able to give several values of c and d for which the sentence is true. They may even offer several pairs without seeing that the values are part of a pattern. More developed responses could be expected to give pairs in a systematic list such that (c, d) could be (9, 1), (10, 2), (11, 3), (12, 4). In this case, are students able to generalise a rule connecting c and d? It might also be possible to probe whether students can give a clear mathematical sense to sentence being true for "*any* values of c and d provided c is 8 more than d". Such responses might make it clear that, for example, fractional or decimal values are possible – or even negative numbers. Being able to derive a correct mathematical generalisation from numerical examples is key element of algebraic reasoning (Carpenter & Franke, 2001; Lee, 2001; Zazkis & Liljedahl, 2002).

A similar question could be constructed for probing relational thinking about subtraction. The value of questions such as these is that they can be given a limited meaning by computational thinkers, but they can only be answered in any depth using relational thinking. A fully elaborated response needs to show that the relationship is determined by the operation as well as the specific numbers involved, and that the sentence can be true for *any* values of *c* and *d* where the given condition is met. This kind of question is likely to be a better predictor of success in dealing with literal expressions.

Conclusion

Students' use of relational thinking to solve number sentences is evident in all three countries by the end of elementary school. The extent of its acquisition varies between countries and between schools. Even where it appears to be strong, there are still many students who seem unable to use it. Those who were consistent relational thinkers on number sentences were more likely to deal successfully with a sentence involving literal symbols and number terms than those who showed only arithmetical thinking on the number sentences. In all three countries, particularly in Years 5 and 6, the majority of this latter group was unable to deal successfully with the sentence involving literal symbols. This group especially should concern teachers. They may obtain perfectly correct answers to number questions through careful use of computational based approaches, but these approaches are clearly deficient when students are confronted with questions using literal symbols where computation will not work. Their inability to use relational thinking means that they are not well prepared to deal with the kinds of thinking – in particular, those involving equivalence and compensation – that they will need in high school algebra. More importantly, one should ask how much better their understanding of number and arithmetical operations might have been in primary school if they had been introduced to and were able to use relational strategies to solve number sentences.

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