Early Notions of Functions in a Technology-Rich Teaching and Learning Environment (TRTLE)

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This paper focuses on notions of function Year 9 students hold as they begin to study functions. As these notions may be fragile, the questions, tasks, and ways of interacting orchestrated by the teacher to elicit depth of understanding, or allow observation of changing notions, are of interest. Extended tasks where students were required to make choices about solution paths provided opportunities for students to develop and consolidate their concept images. Discussion between small groups provided the best evidence of developing and stable conceptions held by students in contrast to written scripts where the strength of these understandings was not evident.

The Function Concept and Student Understanding

The National Council of Teachers of Mathematics technology principle (2000) states, “technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students’ learning” (p. 24) and suggests that all students should have access to technology that can allow higher order mathematical thinking to occur. The role of the teacher is vital in this as it is the teacher who “must make prudent decisions about when and how to use technology and should ensure that the technology is enhancing students' mathematical thinking” (p. 24). Opportunities in *Technology-Rich Teaching and Learning Environments* (TRTLE’s) have opened the door for easy access to the multiple representations of functions. The study of “multiple representations of functions is important in secondary school mathematics curricula, yet many leave high school lacking an understanding of the connections among these representations” (Knuth, 2000, p. 500).

Functions have been the focus of much research in recent years (e.g., Yerushalmy & Shternberg, 2001). The complex nature of functions has resulted in many student difficulties being identified (Knuth, 2000). Janvier (1996, p. 233) argues, “the notion of function conceals a wide range of concepts (so much so that one should more correctly speak of the notions (plural) of functions).” This view is not unique, Dreyfus (1990) suggested that due to its many layers of complexity and related sub-concepts “it may well be one of the most difficult concepts to master and teach of all school mathematics” (p. 122). Tall (1996) describes one purpose of functions to be “to represent *how things change*” (p. 289). He also notes that in practice the functions students experience are “first linear, then quadratic” (p. 298). Many schemas and frameworks have been developed to describe and analyse understanding of functions (e.g., Tall, 1996; Vinner & Dreyfus, 1989). Some incorporate the idea of representations (e.g., Moschkovich, Schoenfeld, & Arcavi, 1993) and the impact of technology use (e.g., Confrey & Smith, 1994). Understanding of functions has been the object of study both for students (Sfard, 1992) and teachers (Chinnappan & Thomas, 2003).

Sfard developed a schema allowing “different facets of the same thing” (1992, p. 60) to be applied to algebraic thinking. When the focus of mathematical thought is the function concept, Sfard’s schema describes the dual nature of the function as needing to be

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understood both as an *object*, that is, structurally, and as a *process*, that is, operationally. Sfard argues that the operational view invariably precedes the structural view. At the structural level the view of a function is of two sets having some relationship or correspondence between them. At the process level a function is some method of determining one value given another. Moschkovich, Schoenfeld, and Arcavi (1993) refine the object-process schema by introducing representations into the schema. They suggest that "from a *process perspective*, a function is perceived of as linking x and y values … From the *object perspective*, a function or relation and any of its representations are thought of as entities" (p. 71). Both perspectives are essential to a full understanding of functions according to many writers (e.g., Moschkovich, Schoenfeld, & Arcavi, 1993; Sfard, 1992). Vinner and Dreyfus (1989) use the constructs of *concept definition* and *concept image* to distinguish between the formal definition held by a learner about a concept and the broader set of images a learner holds about a concept. Vinner and Dreyfus (p. 356) define concept image as "the set of all mental pictures associated in the student's mind with the concept name, together with all the properties characterising them".

Linear Functions and the Notion of Gradient

Leinhardt and Steele (2005) investigated what understandings Grade 5 students can develop about linear functions and the role of classroom discourse in this. The students in this classroom discovered many important ideas including, "that the function rule [2X + 1] is a line" (p. 139), "some students discovered that the graph itself could be used to check … [and] to predict values" (p. 139). Their research suggests that students developed "intuitive ideas of slope and parallel slope" (p. 155) and "recognise[d] patterns in the connections between pairs of pairs [all] quite subtle notions for fifth graders, yet these students generate[d], them spontaneously and they do so publicly" (p. 155). Another study in the primary years, using carefully structured situations, also found that 8 to10 year olds can develop important ideas related to functions (Schliemann & Carraher, 2002). Grade 3 students were able to consider functional relationships; make generalisations, including using mapping notation ($n \rightarrow n + 3$) and n "to represent any value" (p. 255); and make connections between situations and the algebraic, numerical and graphical representations of these. With respect to gradient, Schliemann and Carraher report that third-graders "can start to understand how straight lines in a graph represent the same ratio" (p. 263).

Unlike much research in this area, the students who are the focus in this study are in Year 9, just beginning their study of functions. Students' conceptions of gradient within the study of linear and non-linear functions is the major focus of this paper. In this situation, where knowledge is often, understandably, fragile, the research questions of interest are: "What notions related to the function concept do students have?" and "How do we know what they know?"

Methods

In this paper one Year 9 TRTLE is being considered. Students in this class had their own laptops and TI83/83Plus graphing calculators. A qualitative approach was chosen to provide a comprehensive picture of what was occurring within the TRTLE. A case study using an instrumental approach (Stake, 1995, p. 3) was used. The purpose was to study the case to "understand the phenomena or relationships within it" (p. 171) in order to establish what understandings of function students demonstrated in their early study of function in a

TRTLE. Manifestations of the phenomena being studied, namely student understanding of functions, and the conditions that enabled, promoted, or impeded this were identified. Data sources for this paper were teacher interviews, observational notes, audio and video recording of 3 lessons on a teacher designed task building function notions, and a sequence of 14 lessons on linear function, and documentary evidence (student work including assessment scripts, handouts of teacher presentations, and student task sheets).

Teacher Orchestration: Developing Students' Understanding of Function

Student understanding of function was developed through two main arenas in this TRTLE. First, the teacher, Peter (all names are pseudonyms), developed and implemented a 14 50-minute-lesson unit of work focussed on linear functions. Secondly, the students were involved in a series of extended teacher designed tasks, building function ideas, over the course of the year. In both of these arenas, the teacher integrated technology into his teaching program. Two of the extended tasks, *Cunning Running* and *Shot on Goal*, were implemented prior to the linear functions unit. Both tasks were made accessible to Year 9 students by the use of technology. Students were able to recognise structure across repeated by-hand calculations that were then duplicated using LIST formulae to replicate and extend each set of calculations. Subsequent concatenation of these formulae and their transformation to an algebraic function enabled students to work with functions numerically (LISTS), graphically (plot and function graph) and algebraically (symbolic LIST formula and algebraic function) as shown in Figure 1. Both tasks enabled students to develop their function concept image, and specifically to consider what information was offered by each of the various representations.

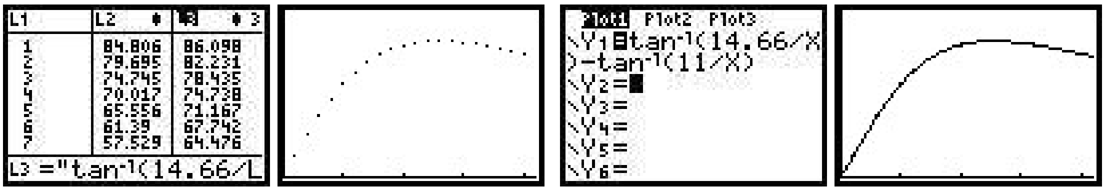


Figure 1. Multiple representations of the *Shot On Goal* function.

Early Notions of Gradient and Optimum Values of Functions

During the task *Shot on Goal*, students were asked to give the positions of the two shot spot distances between which the maximum angle for a shot on a hockey goal occurred. This was in the context of having completed a table of values of angles subtended by the goal mouth at various shot distances from the goal line. Responses to this task allowed insight into students' beginning notion that both discrete functions and continuous functions exist. Ben, for example, considering the function from an object perspective, showed evidence of a developing concept image of a continuous function related to the notion of the maximum value of a function. Ben was searching for where the maximum shot on goal angle occurred for a particular run line.

Ben: Okay so. 9.18, 9.19. So it is between 19 and 20 metres. [Reading from his table of values]
Ken: Why did you say that?
Ben: Because that is 9.19 [at 19m shot spot] and that is 9.18 [at 20m shot spot].
Ken: But then it could be between …
Ben: Could be between 18 and 20.
Ken: It **is** between 18 and 20.

Ben: 18 and 20.
Ken: 18 has a big jump though. [(18, 9.11), (19, 9.19), (20, 9.18)]
Ben: See 18 is 9.11, 20 is 9.18. So between, [pause] the biggest opening is between 18 and … [pause]. The maximum. [pause]
Ken: Ohh, the best angle you can get.
Ben: The maximum angle for the shot on goal [pause] is between 18 and 20 metres, between 18 and 20 metres, two spots—18 and 20.
Ken: 19 and 20.
Ben: It could be there is a higher point between 18 and 19. And it could start dropping again.

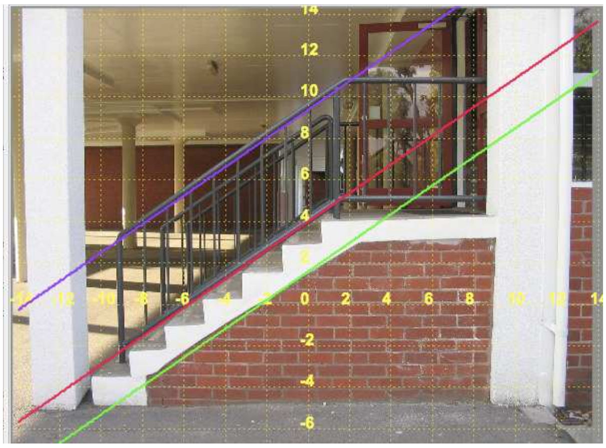
It appears that it was Ken's questioning that resulted in Ben originally expanding his notion of the maximum from being between the two largest calculated angles to a larger set of possibilities. Although Ken initiated the expansion of Ben's concept image, his questioning and Ben's subsequent explanations did not appear to have the same impact for Ken himself. From observations of these and other students attempting several optimising tasks a framework of developing images of the optimal value of a function has been developed (Figure 2). These images are hierarchical however, it is likely that the image held by students is initially fragile and hence they may move between images prior to moving to a stable concept image. In this TRTLE, students were identified at each stage in the framework.

Image A: Students simply see a set of numerical values where the optimal value is obvious – a range of values for the optimum is meaningless if the data are all different as they are discrete values.
Image B: Students have expanded their concept image here to include the graphical representation in addition to the numerical. These students have a mental image of the plot of the function, but still see only discrete values and the highest/lowest data point is the absolute optimum.
Image C: Extending the image of B, students show evidence of considering the situation as represented by a function of continuous values and visualising a "curve" passing through their mental image of the plot of points, however, this image has the curve reaching a maximum/minimum that is one of the data points.
Image D: Students show evidence of considering the optimal point can be at a point other than their discrete values however they consider this to be possible on only one side of the optimal discrete value, e.g., between the two highest/lowest values.
Image E: Students show evidence of recognising that the maximum value of the function could be on either side of the maximum/minimum or at this discrete value.

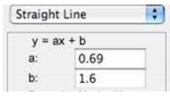
Figure 2. A Framework of images associated with the optimal values.

Linear Functions Unit

Peter changed the way he taught and what he allowed his students opportunities to learn, because within a TRTLE in the linear functions unit, Peter was able to begin the unit in a non-traditional way through the use of the computer application, *GridPic* (Visser, 2004). The use of this software, allowed the students to focus on functions in a new way (Figure 3). Peter (2004) explains "we started with the visual, which *GridPic* allows, [then] with the numerical which they then tried to algebracise to that pattern". He wanted his students to make more sense of slope than simply recalling and applying "rise over run". He felt that his approach using photographs of stairs in *GridPic* developed a deeper understanding. Peter continually emphasised slope as the ratio of the "change in the *y* values" and "the change in the *x* values". Peter's belief that previous teaching had not resulted in what he considered to be understanding, was demonstrated when one student commented, "We learnt it last year as run over rise." Peter responded with a laugh, "It was rise over run, so you didn't learn it at all!" Research findings by Walter and Gerson (2007) support Peter's belief that the notion of gradient as a process, rise over run, does not lead to a full understanding conceptually enabling application and explanation of the concept in a variety of contexts. Observation of Peter's classroom suggests that he was striving to develop in his students, higher order reasoning and understanding that would enable such explanation.



Clicking on a point shows the point on the graph screen and adds the coordinates of the point to an adjacent table.



Selecting a straight line graph and entering values for a and b results in the graph of y = ax + b being displayed. Any picture can be used as the background, or the picture can be hidden with a more standard graphical display presented.

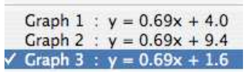


Figure 3. Using GridPic: Ned's three linear functions.

The first two lessons in the sequence saw students using *GridPic* for various purposes including to consider functions in a non-traditional way. Tasks included identifying points and trying to match a line to the stair rails (Figure 3). Conceptions related to function that were raised in the first lesson are shown in Figure 4. All these were orchestrated by the teacher except one (indicated with [S]) which was raised by Ned who was the student operating the computer with the display projected for the class to see.

Conceptions of Gradient

The three lessons forming the conclusion of the 14 lesson sequence were spent on a linear functions task. Students were presented in pairs with a task involving either the cost of installing safe drinking water wells or the cost of running small village health clinics. There were five different versions of the task. Introductory information and details of the elements of Part 1 and 2 of one version of the task are given in Figure 5.

The task was implemented as two consecutive lessons totalling 100 minutes on the Wednesday and a single period on the Friday in the last week of term 2. Twenty-four students were present. Two students were present for lessons 1 and 2 but not the third and a final student was present only for lesson 3. Scripts for 19 students were collected with a script for Ken, being recreated from his audio recording. One pair of students was video recorded with a further two pairs audio recorded. Of particular interest here is students' understanding of gradient as elicited by this task.

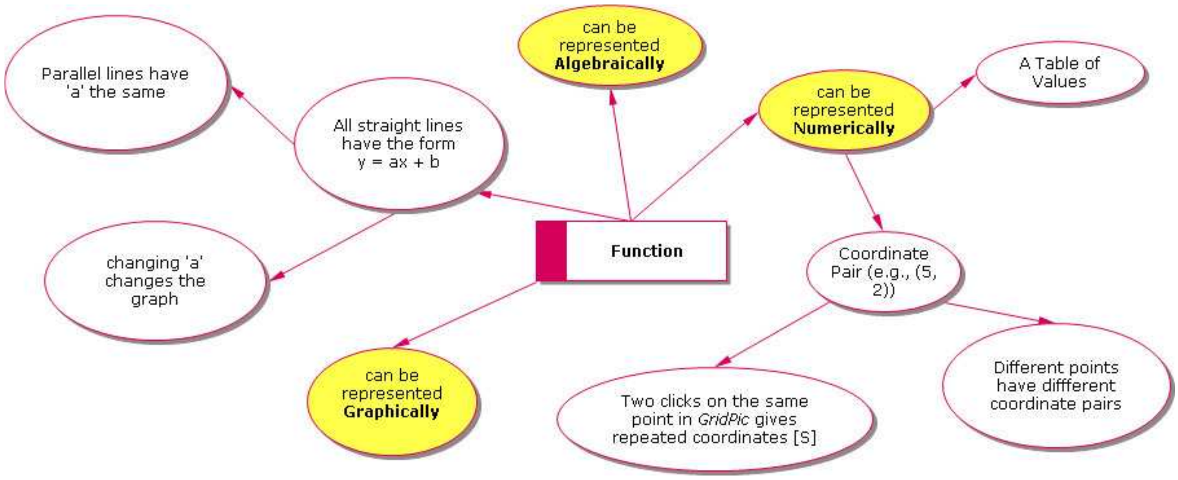


Figure 4. *Webs of meaning: Conceptions related to functions raised during lesson 1.*

A village Health Clinic in Mali

The weekly cost of running a small village clinic at Lake Haogoundou in Mali is a function of a constant weekly value and varies as the number of patients (n) attended. The cost is \$1100 when 50 patients are treated and \$1740 when 90 patients are treated.

Part 1 of the task required students to: Draw a plot showing a linear relationship for an appropriate domain. Identify the relationship decoding from text. Identify the domain, and dependent and independent variables. Using a linear rule C(n), find C given n. Write the linear relationship as an algebraic rule. Find n, given C.

Part 2 of the task [Functions for two other clinics are given: Bamako: COST = 390 + 17.50 × number of patients, Timbuktu: COST = 115 + 19.75 × number of patients] required students to: Find the costs, C_B and C_T, given n = 50, 60, … 200 recording in table of values. Focusing on one specific value of n state which cost is cheapest, C_B or C_T. Calculate $|C_B - C_T|$ for this value of n; Determine the value of n when C_B becomes lower than C_T. Construct a table to support this result. Explain how the table of values supports these ideas. Graph the 2 functions over an appropriate domain. Identify rate for C_T. Identify rate for C_B.

Figure 5. The Linear Functions Task.

Determining the gradient. In Part 1, students were required to determine the equation of the given function, hence this involved the calculation of the gradient. All students demonstrated they were able to calculate the gradient as evidenced by their written response to this part of the task. However, two students, Kit and Rani did not identify the equation of the line that passed exactly through the two points given. Instead, when completing the table of values prior to the determination of an algebraic rule, they used estimates for function values from their graph. However, their responses indicated they knew how to calculate the gradient. Althought from a purely function point of view, one could argue that these students were unable to calculate the gradient correctly, from a modelling perspective, one can equally well argue that the student moves were perfectly valid. Peter, in fact, encouraged them to determine their equation from their graphical representation of the situation.

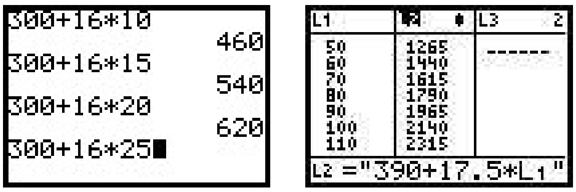
Like most of the students, Ben and Ken, for example, considered the gradient as a ratio to be calculated.

Ken: Well we have two points.
Ben: Yeah, so.

Ken: 1100 minus.
Ben: No, 1340 - 1100. [Calculates (1340 – 1100)/(170 – 90) = 240 ÷ 80]. 240. Divided by 80. So the gradient is three.

Use of the schema of Moschovich, Schoenfeld, and Arcavi (1993) showed that all students considered the gradient as a process, not an object, and most students operated in the numerical representation. For these students a connection was made between the numerical and algebraic representations, as a process was undertaken on numerical objects to determine the value of the gradient or “a” in the function equation y = ax + b. Kit and Rani, however, operated across three representations. Having represented the initial numerical information graphically, they were asked to complete a table of values. They completed this by reading values from their graph and subsequently used these values to determine the gradient. They were the only pair to do this in determining the gradient although they and two other students used the graphical representation to determine the y intercept. The majority of the students continued to work in the numerical representation to determine the value of b in y = ax + b, using a substitution method.

Opportunities for using the gradient. In Part 2 of the task, students were initially required to complete a table of values for two given functions. This could be completed in a number of ways using the HomeScreen of the graphing calculator to enter each calculation individually as undertaken by Kit and Rani (Figure 6a), or using the LISTs in the graphing calculator as undertaken by Kate and Meg as shown in Figure 6b.



(a) (b)

Figure 6. Completion the table values using (a) the HomeScreen and (b) LIST formula.

A third method, used by Ken and Ben, made use of the gradient employing what is described by Walter and Gerson (2007) as “slope as an additive structure” (p. 227). Ben was quick to see that he could use the gradient to complete the table of values more efficiently. Ben recognised that as the number of patients was given in increments of 10, the corresponding cost values should increase at a rate of 10 times the gradient, that is 140. Having determined the costs for 50, 60, 70, and 80 patients as 855, 995, 1135, and 1275, Ben stopped when he noticed the constant increment.

Ben: Hang on, hang on. Stop for a second, ha, 995 – 855, 140. So I think it is just going up by 140. So 1415 + 140 [Adding 140 to Ken’s last calculation for y(90).]
Ken: What? How did you work that out? It is not going up by 140.
Ben: Yeah, it is.

The students then continued using this additive method for further calculations in the task.

Identifying the gradient. Students were asked to identify the cost for treating each additional patient at the two health clinics being considered. The parallel question for the Water Well versions of the task required students to identify the cost for each metre of well depth drilled. Analysis of student scripts showed that 15 students recorded a correct answer to the questions. Only two students recorded a correct method. Another 11 students recorded no method, so unless other evidence is available it is difficult to infer what understanding these students have. One did not attempt this part of the task. An incorrect solution recorded by Kit and Rani for the first clinic was the result of $y(10) \div 10$, but the question for the second clinic was not answered. Kit originally recorded the cost for one patient, but this was subsequently crossed out. Di and Ann, working together, both recorded the cost for one patient, not appearing to realise that the gradient was required.

It is important to note here that during the final lesson on the task, each pair of students was expected to discuss their work with a second pair who had attempted the same version of the task. This provided students with the opportunity to check and discuss their results. Recordings of these discussions shows significant differences in student understanding of the interpretation of this question. When attempting these questions for the task (Figure 3), the transcript of Kate and Meg shows clear evidence of developing understanding of the concept of gradient. For Kate and Meg, their first function [$C_T = 115 + 19.75n$] was for the cost of treating patients at a clinic in Timbuktu. Initially, the pair was unsure what the question was asking. Meg began by finding the cost for one patient. She appeared to have some sense that they were finding the gradient but this knowledge was fragile.

Meg: What have I got? I think you just, I am going to put in just one [hesitantly].
Kate: Mmm. And then?
Meg: Okay, 19.75 × 1 + 115, [pause] whatever.
Kate: How come you are doing it times 1?
Meg: Umm, because when you find each additional patient after, like from, you go up by one.
Kate: Oh yeah.
Meg: It is hard to explain. Each time it goes up by. Each time it adds on to the 115 [fixed cost].

Later, Meg suggested that she was now confident that they are being asked for more than the cost for one patient.

Meg: Wouldn't it be 20 [approximate difference between C(2) and C(1)]?
Kate: What?
Meg: I thought it asked to do one, but to do each additional one. So what if you times it my two and then take away what you got for one. Like then it is about 20. It will go up by 20 each time you treat somebody.
Kate: Why 20?
Meg: Because you times. If you treated 2 patients you add, yeah you get, well, [how] much it would cost and then you take that away from one [C(1)]. No take 1 away from that. [C(2) – C(1)].
Kate: Wait, what are you saying? You go 19[.75 × 1 + 115] [pause]. Yeah , you do that right?
Meg: Yep, yeah.
Kate: Then what?
Meg: Then there is like one, and then there is like two. And there is like the difference. I think the difference there is like 20. Because if this here is 134.5 [C(1) = 134.75], and this one here is like 15.5, no 14.5. Wouldn't you just take them away? [C(2) = 154.50]
Kate: [incoherent] It is 154.5.
Meg: That is how many if you treat two patients, that is how much it costs.

Kate showed evidence of understanding what Meg was saying, as she then suggested that for a third patient the cost would be an additional $20 compared to the cost for two patients. The pair then proceeded to calculate similar values for the clinic at Bamako. An error in their subtraction, which was identified as they checked their results, led to their recognition that they had found the gradient.

Meg: How come is it 19.75? Oh my god! That is the gradient it goes up by each time. So it is 19.75.
Kate: That is the gradient?
Meg: Yes, the gradient goes up by, each time it goes up by. Oh my god!!

Kate: Yeah, der, because it is ‘a*x*’. [laughs] Der. [laughs]
Meg: Each additional patient costs 19.75. Because that is what the gradient is. Because each, the gradient is, whenever you go up by one, no whenever you go across [referring to a visual image of a graph]. When you treat one more patient.
Kate: Yeah I know, I get it.
Meg: So it is 19.75?
Kate: For each additional patient. So it is not that? [referring to $134.75 calculated previously].
Meg: No, that is just how much it costs to treat one patient.
Kate: So this one [referring to the similar question focussed on the second clinic] would be the same thing. This one would just be 17.50, I guess.
Meg: Yeah.

For Kate and Meg in particular this sub-task provided the further opportunity to develop understanding of the notion of gradient incorporating this additive structure. Fortunately, their discourse was captured via the audio recording, as the fragility of their knowledge was not evident on their written scripts.

In contrast, for two other students who came together during the final lesson, this sub-task seemed trivial as their notions of gradient were stronger.

Amy: [Reads] “What is the cost for treating each additional patient at Malange?”
Ben: 14.
Amy: Yep.
Ben: And then 12.50.
Amy: Yeah. Gee that is intelligent having to figure that out!
Ben: Yeah I know. [Facetiously] That was the hardest question!
Amy: Yeah.

Scripts for the students discussed here, for this particular sub-task, show little evidence of either the fragile yet emerging understanding of Kate and Meg, or the stable understanding of Ben and Amy. Only through access to their dialogue has this become evident.

Discussion and Conclusions

The students in the TRTLE that was the focus of this study demonstrated both process and object perspectives of various function notions. Both optimisation tasks and the linear functions task provided opportunities for, and evidence of, students thinking about and making connections between functions in each of the numerical, graphical, and algebraic representations. The fragility of these notions was evident at times through the choices of representations selected in solving tasks. Both the schema of Moschkovich, Schoenfeld, and Arcavi (1993) and the concept image construct of Vinner and Dreyfus (1989) proved valuable in exploring students’ developing notions.

The nature of the tasks presented to the students enabled them to make choices, including when to use technology and when to use by-hand methods. Additionally, when choosing to use the available technology, students had to make choices as to methods and representations in which to operate. These opportunities for choice at times allowed students to demonstrate their stable understanding of particular notions related to function. At other times, these same opportunities allowed students, in discussion with others to develop new understandings or to challenge fragile understandings and consolidate these.

The extended tasks where students were working relatively independently of their teacher and placed in the position where they made choices about solution pathways were particularly valuable in providing students with opportunities to develop and consolidate understandings of essential notions related to function. The expanding concept image enabled by engagement with the tasks led to broader, deeper, or more stable conceptions of the function concept. However, evidence of the strength of student understanding was more likely to be detected through attending to classroom discourse, at a private level between two or three members of the TRTLE, rather than through written work as little trace of this was recorded by students on their task scripts.

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