Teaching and Learning by Example

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The mathematical problems, tasks, demonstrations, and exercises that teachers and students engage with in classrooms are, in general, specific instantiations of general principles. Indeed, the usual purpose of such examples is to illustrate those principles and thus facilitate their learning. With this in mind, it is clearly important for teachers to be able to choose or design suitable examples, to recognise what is offered (or afforded) by particular examples, and to know how to adapt an already existing example to better suit an intended purpose. Although writers of textbooks and other teaching resources also need these skills, it is ultimately the teacher who puts the examples to work in the classroom. Teachers' choice and use of examples is indicative of their pedagogical content knowledge (PCK)—the complex amalgam of mathematical and pedagogical knowledge fundamental to teaching and learning—and reflects their understanding of the mathematics to be taught and how students can be helped to learn it. This paper examines some of the issues associated with example use and how it is informed by and can inform us about PCK.

When a mathematics teacher asks a class to find the solutions of $x^2 - 5x + 6 = 0$, an observer may already have an idea about the point of the exercise. The task appears to be about solving equations—more specifically, quadratic equations. Beyond this, however, some contextual information is needed in order to understand fully the teacher's purpose in choosing that particular example. What if the next problem assigned is to find the solutions of $x^2 - 2x + 5 = 0$? Does this tell us anything? The two problems do not appear very different structurally, so why assign both? How are the two problems the same and different? What more does the second example tell us about the teacher's learning intentions?

This scenario highlights a number of issues. First, the teacher's purpose in using the tasks most likely is *not* to solve the specific problems but to teach more general principles. The actual solutions to the specific equations $x^2 - 5x + 6 = 0$ and $x^2 - 2x + 5 = 0$ are not of interest, but the teacher is likely *very* interested in highlighting conceptual issues such as equation-solving methods and the nature of solutions. Second, the purpose of an example is always context dependent. In this case, the presence of the second problem suggests that the focus of the learning activity might be on the fact that some equations do not have real solutions. Third, a particular example may be used to exemplify different things. For instance, the intended purpose for solving the equation $x^2 - 5x + 6 = 0$ might be factorising, completing the square, using the quadratic formula, or highlighting the fact that an equation can have more than one solution. Finally, for an observer to determine (or hypothesise about) the purpose of the examples requires mathematical knowledge. More significantly, however, the *teacher* had to know what mathematical ideas she wanted to convey and, with this knowledge, needed to be able to design or choose examples to suit her purpose.

Although this illustration comes from the secondary mathematics curriculum, the principles apply more broadly, including to primary mathematics teaching, the focus for the research reported here. Investigating these issues closely involves a consideration of what constitutes appropriate teacher knowledge, how to examine opportunities inherent in classroom activities, what is meant by "example", and how examples can be used.

Background

Pedagogical Content Knowledge

Before narrowing the focus to that part of teaching that involves example choice and use it is useful to briefly examine the broader domain of *pedagogical content knowledge* (PCK). Shulman's 1986 introduction of the term highlighted the fact that teacher knowledge—and resultant teacher effectiveness—depends on more than discipline content knowledge alone. He identified many of the facets of knowledge that contribute to PCK, including knowing what models and explanations support learning, understanding typical student conceptions, and recognising what makes a task complex or easy. These have now gained the attention of many researchers who have examined the nature of this knowledge in more detail. Other aspects of PCK include knowledge of connections among and within topics (e.g., Askew, Brown, Rhodes, Johnson, & Wiliam, 1997), deconstructing knowledge into key components (e.g., Ball, 2000), content knowledge (e.g., Kahan, Cooper, & Bethea, 2003), knowledge of representations (e.g., Leinhardt, Putnam, Stein, & Baxter, 1991), and Profound Understanding of Fundamental Mathematics (PUFM) (e.g., Ma, 1999). Lampert (2001) highlights the complex interplay among aspects of PCK in the classroom milieu. Drawing on this work, Chick, Baker, Pham, and Cheng (2006) developed a framework for pedagogical content knowledge (see Appendix 1). The framework attempts to identify the key components of PCK, how they are evident in teaching, and the degree to which both pedagogical and content knowledge are intertwined (see also Marks, 1990).

Everything that a teacher does—planning lessons, implementing them, responding to what arises in the classroom, interacting with students—involves one or more aspects of PCK. A lesson on the numeration of decimals, for example, might involve the decision to use a particular model to illustrate the concepts. This requires knowledge of different models and what they offer, recognising that their strengths and weaknesses depend on their *epistemic fidelity* (see Stacey, Helme, Archer, & Condon, 2001), that is, the capacity of the model to represent the mathematical attributes of the concept effectively. Having chosen the model, the teacher then has to use it appropriately in the classroom, recognising the students' present levels of understanding, developing appropriate explanations, and finding ways to respond to students' uncertainties and questions. The tasks that are then set in order to consolidate understanding or to foster its further development also reflect the teachers' PCK, since they should match the lesson's learning objectives.

Affordances and Didactic Objects

Considering tasks and how useful they might be in the classroom requires an evaluation of what they have to offer. Gibson (1977) introduced the term *affordances* to refer to the uses perceived for an object by a potential user. So, for example, a chair affords uses as a seat or a bookshelf but, at first, may not seem to afford a use as an umbrella. That said, however, observing a gorilla holding an upturned chair over its head in the rain reveals that, in the gorilla's perception, "rain shelter" *is* one of the affordances of a chair, and, thus, becomes an affordance of the chair for the observer now that the observer has perceived it too. This emphasis on the "perceived" uses is problematic, especially for some of the issues considered here, because in teaching there are many opportunities and examples that have the potential to be applied in pedagogically useful ways, and yet are not because the teacher does not perceive the opportunity. As a consequence, the term *potential affordances* is used to refer to the opportunities that are inherent in a task or lesson. A teacher may well be aware of some of them—indeed, awareness of these potential affordances is usually evident in how the task is used—but the teacher may not necessarily be aware of all of them, or even the "best" of them. Furthermore, in the unscripted world of the classroom, some of these opportunities may not come to fruition because of other interfering factors; as Anne Watson writes, learning environments involve "a complex interplay between what *could be* possible, what *is* possible, and what *is seen as* possible" (Watson, 2003, p.37). A teacher's PCK influences the degree to which she identifies the potential affordances in tasks and activities, makes pedagogical choices that allow her to offer desirable affordances in the classroom, and then finds ways of making those affordances give rise to effective learning.

Thomson (2002) talks more specifically about the role of discussion and usage in the learning process, and uses the phrase *didactic object*

> … to refer to "a thing to talk about" that is designed with the intention of supporting reflective mathematical discourse. … [O]bjects cannot be didactic in and of themselves. Rather, they are didactic because of the conversations that are enabled by someone having conceptualized them as such. (p.198)

This has relevance to models and representations, and, of course, examples. To illustrate this for models, note that although multi-base arithmetic blocks (MAB) are conventionally used to model base 10 numbers—especially units, tens, hundreds, and thousands—they can also be used to model decimal numbers. To do so, however, requires a reconceptualisation not only for the teacher, but also for the students. The MAB blocks afford the opportunity to model decimal fractions, but the reconceptualisation is needed to turn them into a didactic object. A whole new set of conversations must be evoked by the teacher in order to use MAB in this way, at the same time taking account of the epistemic fidelity issues (again, see Stacey et al., 2001). An example has the same capacity, potentially affording many things but delivering none until conceived as a didactic object. "Find the solutions of $x^2 - 5x + 6 = 0$" could illuminate many concepts, but its purpose must be identified by the user and then utilised in such a way that the desired concepts become apparent.

Examples

The meaning of "example" has, so far, been assumed as understood. It is necessary, however, to define it. For the purposes of this paper an example is a specific instantiation of a general principle, chosen in order to illustrate or explore that principle. This covers the usual sense of "example", such as a teacher making a point by giving a specific illustration (e.g., "eight is an even number because it can be written as two times a whole number") or demonstrating a solution procedure (e.g., a calculation using the long multiplication algorithm). It also covers assigned exercises and extended tasks.

Bills, Mason, Watson, and Zaslavsky (2006) give an extensive overview of the history of example use and the role of examples in learning theories. Ball (2000) highlights how a particular task needs to be examined by the teacher to determine what it offers students, and then discusses the issue of deciding how to modify the task to make it easier or simpler, or to make it illuminate particular concepts. Watson and Mason (2005, 2006) highlight the way in which changes to examples can highlight different concepts, and also show that getting learners to construct examples provides rich learning experiences. In fact, the situations discussed in the early chapters of their 2005 book show two significant aspects of examples. Although their primary thesis concerns examples constructed by students and how these develop mathematical understanding, in most cases these examples would not be generated without an appropriate task assigned by the teacher. Some of these tasks are quite open (e.g., "Construct a data set of seven numbers for which the mode is 5, the median is 6 and the mean is 7", p.2). If the teacher's intention with the task is to have it illustrate a general principle, notwithstanding that the students develop the specific instantiations, then it is argued that this makes the task an example too—perhaps in a "meta" sense, but an example nevertheless. Indeed a task may reflect more than one level of example-hood. A teacher may, for instance, select the "pizza" model for fractions—with the pizza exemplifying a fraction—and then ask students to show that $^1/_4$ and $^2/_8$ are the same—with the choice of $^1/_4$ and $^2/_8$ intended to exemplify general issues associated with equivalent fractions.

For all that a specific example may be an instantiation of a general principle, one of the key concerns in example use is to ensure that the general is revealed out of the particular. This requires teachers to identify the important and unimportant components of the example that illustrate the generality. Bills et al. (2006) cite a case from the work of Rowland and Zaslavsky illustrating how variation in some digits in the subtraction problem 62-38 still allows regrouping to feature, but that other choices "ruin" the problem for that purpose. Watson and Mason have adapted an idea of Marton (cited in Watson & Mason, 2005), *dimensions of possible variation*, to discuss ways in which an example's scope can be varied. Skemp (1971, pp. 29-30) talks about the role of *noise* in examples, and that identifying the general principle requires the learner to distinguish the salient features from the extraneous. One key implication of this is that teachers' example choices must allow the relevant features to be detected through the noise (although Skemp points out that some noise is important). Since there are often many variables and features in an example, choosing the appropriate instantiations is critical, and requires adequate PCK.

Returning to the framework for PCK (Appendix 1), all aspects of PCK can influence example choice and use. Of particular significance are (i) the underlying content-related aspects—such as PUFM and knowledge of connections and representations; (ii) knowledge of student thinking—both current and anticipated, together with knowledge of likely misconceptions; and (iii) the capacity to assess the cognitive demand of a task.

Bills et al. (2006, p.138) suggest that there is a scarcity of research on teachers' choice of examples. Zazkis and Chernoff (2006) describe a situation where a researcher taught a student about prime numbers through choosing strategic examples, with the teaching situation such that examples had to be generated spontaneously rather than being planned in advance. This clearly relied on the researcher's deep understanding of prime and composite numbers and the ability to construct examples that were appropriate for the student's needs. Zaslavsky, Harel, and Manaster (2006) examined the mathematical knowledge brought into play by a teacher introducing Pythagoras' Theorem to students on two different occasions. On the first occasion the cases chosen were intended to build up to the general result and reflected the teacher's understanding of geometrical configurations that are useful for Pythagoras' Theorem. On the second occasion the physical constraints of the way she had set up the examples—needing all sides to be integers—reduced the number of examples that could be given and may have affected the students' capacity to see the entire generalisation. Little has been done to investigate more specific aspects of PCK; this is part of the purpose of the present study.

Finally, it should be noted that many researchers have actually used examples to probe PCK. Hill and colleagues (Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004) have used multiple-choice questions that require teachers to examine a situation—a specific instantiation of a general scenario, involving a particular mathematical problem—and identify appropriate content- or pedagogically-based responses. Watson, Beswick, and Brown (2006) used a particular fraction/ratio problem to probe teachers' content knowledge, with follow-up questions investigating teachers' knowledge of students' likely thinking, including misconceptions, and their possible approaches for teaching the topic or remediating difficulties. The project from which the present research is drawn also used teaching situations based on specific examples to probe different aspects of teachers' PCK (see Chick & Baker, 2005a; Chick, Baker, et al., 2006; Chick, Pham, & Baker, 2006). In all cases the examples used were designed carefully in order to reveal general rather than specific aspects of the levels of PCK held by the teachers.

The Focus of this Paper and the MPCK Project

The current study considers some of the examples used by upper primary teachers. The intention is to examine the affordances inherent in the examples, and the way in which the teachers implement them to turn them into didactic objects. This examination provides insights into the teachers' PCK, and what needs they may have for developing it, particularly in regard to example choice. Although the examples are from the primary curriculum, it is anticipated that there are general principles that apply for teachers of other age groups.

The data for this study were collected as part of the ARC-funded Mathematical Pedagogical Content Knowledge project. This project involved fourteen Grade 5 and 6 teachers who volunteered to participate over a one- to two-year period. Part of the project's purpose was to examine teachers' PCK and how it is enacted in the classroom. A questionnaire and follow-up interview were used to gather initial data, and then pairs of lessons were observed and video-taped. The two lessons were on the same topic and conducted consecutively, with the teacher nominating the topic for observation. Up to four such pairs of lessons were recorded for each teacher. During the lessons the video-camera focused on the teacher, and the teacher's words were recorded via a wireless microphone that was sensitive enough also to record some student utterances. Field notes were also made. Following each pair of lessons, the teacher was interviewed about the original plans for the lessons, perceptions of successes and difficulties, changes and adaptations made, and future follow-up plans.

Several of the pairs of lessons involved fractions, and these lessons were subjected to a "content analysis" approach (Bryman, 2004), in which individual examples that arose in the classroom were identified, according to the definition of "example", and then categorized according to the way in which the teacher used it. This identified, for instance, whether the example was used as a teacher demonstration, or as a student task; or whether the example focussed on conceptual or procedural matters. From this data, and from data from three other pairs of lessons on other topics (probability, and measurement) several illustrative cases were selected to allow comparisons among the ways in which tasks were used, the affordances they offered, and the PCK involved. The purposeful selection of these cases makes them what Bryman (2004, p. 51) calls *exemplifying cases*, which are used for the purposes of a multiple-case comparative study.

There are, of course, some caveats about what can be learned from such a research design. Although information about teachers' intentions was obtained from the post-lesson interviews, these interviews were wide-ranging and did not always focus on examples per se. Consequently the teachers' purposes have, at times, been inferred from their implementations and classroom actions. Furthermore, it is easy, as an outside observer with the benefit of repeated video viewings, to see alternative options that teachers might have utilised to good effect. It is, however, important to acknowledge the complex milieu of the classroom, the speed with which some decisions must be made, and that, in these cases, mathematics is not the only area of the curriculum that primary teachers must teach.

Three Sets of Examples

This section describes three sets of examples that highlight important issues associated with example choice, affordances, and PCK. As explained earlier, the examples were purposefully selected from the lessons of eight of the MPCK teachers (names are pseudonyms), from nine of their pairs of lessons. The examples were chosen for what they illustrate qualitatively rather than to reflect any quantitative assessment about either the types of examples used in general or by a particular teacher. The scale of the examples varies, ranging from an assigned computational exercise through to an extended problem that the teacher utilised to illustrate a wide range of mathematical concepts. The pedagogical implications—such as the affordances offered by the examples described, and the PCK evident or missing in the choice and implementation of the examples in the classroom—are also examined.

Fractions

Six of the teachers presented pairs of lessons on fractions. In some cases their focus was on the meaning of a fraction, whereas in others they addressed fraction operations. In the majority of these lessons the teachers used many "small" examples, usually illustrations of particular fractions or exercises for students to solve. A range of these are presented here to show what examples were chosen and how the teachers used them, with discussion on what the examples might have afforded and what PCK was evident.

Cake halving. Meg used a square cake and repeatedly halved it, emphasising that the cake is the "whole" and remains the same quantity, but that the pieces were getting smaller. She also clarified the terms numerator and denominator. A student wrote the associated unit fractions on the board, finishing with $^1/_{32}$, and Meg emphasised that as the pieces get smaller the denominator gets bigger.

The idea of "cake cutting" has the potential to model almost any fraction, not just those with a power of two for a denominator nor just unit fractions. Meg's repeated halving allowed students to see some atypical primary school fractions, such as $^1/_{16}$ and $^1/_{32}$, but omitted many other unit fractions. Furthermore, her emphasis on unit fractions allowed a focus on the relationship of the denominator to the size of the piece, but prevented a deep examination of the meaning of the numerator. Although there is no evidence that this caused problems for these students, a well-known misconception is that students will, for example, regard $^2/_5$ as bigger than $^6/_7$ because fifths are bigger than sevenths. Meg may not have been aware of this particular misconception, or, if she was, may not have seen that although her emphasis on the relationship between the denominator and the size of the pieces was important it had the potential to lead to such a problem. Finally, the emphasis on halving appeared to interfere with later examples involving thirds and fifths.

Aero bar. Irene began her introduction to fractions with a KitKat chocolate bar, which allowed her to talk about quarters and emphasise the meaning of numerator and denominator. She also used a piece of paper torn into four pieces to illustrate the importance of having equal parts. Her next example used an Aero chocolate bar, which has seven pieces. She broke off three pieces and asked what fraction would represent how much she had. This example allowed her to illustrate sevenths, a denominator different from the familiar halves, quarters, and thirds. She also pointed out that sevenths are difficult to show with the "pizza" model of fractions.

Irene's choice of chocolate to model fractions suggests knowledge of how to "get and maintain student focus". In addition, by beginning with the four-piece KitKat she could model the familiar quarters, and then use torn paper to emphasise the importance of equal pieces, which had been implicit rather than explicit in the chocolate bar. The KitKat example provided an appropriate segue into the Aero bar, which allowed a "real world" example of sevenths, and Irene also emphasised the role of the numerator. There is a disadvantage in using the two different chocolate bars in that they are not suitable for making comparisons of quarters and sevenths; nevertheless, the chocolate bar models were suitable for the purposes to which Irene put them.

Smarties. After an initial review of fraction terminology and the use of a circle divided into three unequal pieces to emphasise the importance of equal parts, Jill used discrete materials rather than continuous materials to reinforce fraction notation. Students counted the numbers of each colour in small boxes of Smarties, and expressed this as a fraction of the total number of Smarties in the box. They also had to create a fraction strip on grid paper to show the fractions obtained, by dividing the strip into equal parts representing the total number of Smarties and then colouring in the relevant proportions. Unfortunately this model then caused problems when Jill tried to illustrate addition of fractions with the same denominators. She used an example of one person having 12 out of 14 orange Smarties and a second person having three out of 14 orange Smarties and added these as fractions to get $^{15}/_{14}$ (since there is a "common denominator"), before she turned this improper fraction into a mixed number. The problem here, however, was that the situation implies that there were, in fact, 28 Smarties involved. Jill acknowledged that there were actually two boxes of Smarties but told students to treat them as one box.

In theory, at least, the box of Smarties can be used to model fractions, but great care needs to be taken about identifying the "whole". Jill did not give this concept enough emphasis, with the added difficulty that the number of Smarties per box can vary. Jill knew about the latter problem and attempted to address it, but the former issue made modelling fraction addition difficult. In this case, the model/example was inadequate or did not have the level of epistemic fidelity needed to deal successfully with addition of fractions, despite the fact that it was suitable for simply representing fractions.

Fraction wall. Meg used the well-known "fraction wall" idea, and asked students to fold equal length strips into different numbers of parts. Obtaining halves, quarters, and eighths was easy, especially after the earlier cake-halving demonstration. Thirds were a little harder to fold (and some students anticipated that she would ask for sixteenths next), and then when Meg asked them what fraction they could find next, many students suggested fifths, whereas Meg had been thinking of halving again to get sixths. Fifths required even more adeptness at folding, and in the end Meg and some of the students resorted to measuring and calculating the lengths, a task made easier by the fact that the strip was 20cm long. Students did tenths next, and Meg made a conscious decision not to tackle sevenths because of the challenge of finding a strategy for folding the paper into seven. This meant that the students' fraction walls had all the fractions up to eighths and tenths, with the exception of sevenths and ninths.

Since the fraction wall model for fractions uses strips of equal width to build up a wall, the fractional parts are represented both by area and by length. It is a powerful model for comparing fractions, and can also highlight equivalent fractions. Meg's chosen sequence of fractions to make (halves, quarters, then eighths; thirds, then sixths, fifths and finally tenths) echoed her focus on halving as implemented with the cake-cutting activity earlier in the same lesson. There was no detailed discussion, however, of how halving the thirds gives sixths, thus missing an opportunity to strengthen connections between the ideas of halving and doubling. The omission of sevenths and ninths, which Meg acknowledged as being a consequence of time constraints and the difficulties of folding, may have reduced the students' capacity to generalise the fraction concept from the examples given.

Comparing fractions. Lisa had previously done work on equivalent fractions, which provided a foundation for her two lessons on comparing fractions. She began with a pizza comparison, asking students to decide who ate more if one person ate half a pizza and the second person ate four pieces of a pizza that had been cut into ten pieces. Students then had to generate fractions using a deck of cards, by selecting pairs of cards to generate the numerator and denominator of a proper fraction, and then comparing two fractions thus obtained. This led to some challenging problems, in one case involving twelfths and sevenths, which caused difficulty for some students. Prior to the second lesson she asked students to compare $^2/_5$ and $^1/_3$ for homework, and in the second lesson had students show how they had used equivalent fractions to make the comparison. She also showed how the equivalent fractions could be modelled on a fraction bar, giving a very careful discussion of how the fifths on a fraction bar could be turned into fifteenths by dividing each part into three.

Lisa's pizza consumption example provided a relatively simple context for looking at comparison of fractions and equivalent fractions, where one denominator was a multiple of the other. Her use of a deck of cards for generating fraction comparison problems introduced a random element to the tasks, and meant that she lost control of what kind of denominator relationships would arise. It is not clear that this was because she did not realise that denominator relationships might be important, or that the task, as designed, would affect them. The consequence was that some students had to grapple with quite difficult comparisons (such as twelfths and sevenths), which may have been too cognitively demanding for them. On the other hand, the choice of $^2/_5$ and $^1/_3$ for the homework task was more manageable, and afforded the opportunity to relate the problem situation to both the equivalent fraction calculations and to a model used to represent them. The choice of values is particularly good for this purpose: the two fractions are sufficiently close that comparing them demands an equivalent fractions strategy, rather than being obvious through visualisation; the values for the denominators make the calculation and representation of the equivalent fractions achievable yet still suitably cognitively demanding for the students; and the conceptual connections can be highlighted.

Exercises with fraction operations. The lessons that focused on fraction operations had a strongly procedural rather than conceptual orientation. Frank's lesson was purportedly a revision lesson, focusing on all four of the fraction operations. He used the example $^1/_6 + {^3/_6}$ to illustrate addition of fractions with the same denominator, without commenting that $^3/_6$ is, in fact, $^1/_2$, or that the final answer of $^4/_6$ can be simplified as $^2/_3$. A later exercise for students was $5^1/_2 - 2^7/_{12}$ which Frank expected students to solve by converting the mixed numbers to improper fractions and then finding common denominators if necessary. When one student explained that she had subtracted the whole numbers first, found an appropriate equivalent fraction for the half, and successfully regrouped after realising that $^7/_{12}$ could not be subtracted from $^6/_{12}$, Frank's response was to suggest $5^1/_2 - 2^7/_{19}$ as an example that might be difficult to attempt using such a strategy, implicitly privileging the "convert to improper fractions" method.

A second teacher, Brian, provided students with some exercises for converting from mixed numbers to improper fractions. There were four examples written on the board, $1^4/_{10}$, $7^3/_4$, $5^3/_6$, and $8^9/_{12}$, with three not in their simplest form. His emphasis was on the procedure for converting to improper fractions. The non-simplified nature of the fractions was not discussed, either before or after the conversion.

Both Frank and Brian demonstrated sound procedural knowledge. The focus, however, seemed to be on one concept at a time, ignoring other concepts that were evident in the example, as evidenced in Frank's $^1/_6 + {^3/_6}$ addition problem and three of Brian's mixed numbers problems, where the concept of equivalent fractions was overlooked. Here connections among concepts were *not* being established or reinforced; each process—equivalent fractions, operations with fractions, converting among forms—appears to exist in isolation.

Frank's impromptu construction of the example $5^1/_2 - 2^7/_{12}$ was intended to illustrate a situation where it might be difficult to subtract using the fractions in their mixed form rather than converting to improper fractions. Although it made the denominators harder to work with, the resulting example was, in fact, easier to solve using mixed numbers, given that the new choice of numerators actually eliminated the need to regroup. This suggests that whereas Frank could determine some of the cognitive demand of a problem, he could not quickly work his way through the consequences for the example in its equivalent form. In particular, he could not identify which were the salient pieces of the example to vary.

Probability

The next example, first discussed by Chick and Baker (2005b), comes from the topic of probability. Irene, an experienced teacher, and Greg, who was in only his second year of teaching, were Grade 5 teachers in the same school. They had chosen to use a spinner game worksheet activity suggested in a teacher resource book (Feely, 2003). The spinner game used two spinners divided into nine equal sectors, labelled with the numbers 1-9. The worksheet instructed students to spin both spinners, and add the resulting two numbers together. If the sum was odd, player 1 won a point, whereas player 2 won a point if the sum was even. The first player to 10 points was deemed the winner. Students were further instructed to play the game a few times to "see what happens", and then decide if the game is fair, who has a better chance of winning, and why (Feely, 2003, p. 173). The teacher instructions (Feely, 2003, p. 116) included a brief suggestion about focusing on how many combinations of numbers add to make even and odd numbers but did not provide any additional direction. The “example” in this case is the spinner game in its particular configuration.

Before examining what the teachers did in the classroom, it is informative to look at the affordances of this example. Careful consideration reveals that it affords worthwhile learning opportunities associated with sample space, fairness, long-term probability, likelihood, and reasoning about sums of odd and even numbers. The significant issue here, especially in the absence of explicit guidance from the resource book about *how* these issues can be brought out, concerns the choices that teachers make when implementing this activity; especially in terms of what they allow it to exemplify. To add to the complexity of what is already a conceptually rich example, the configuration of the spinners generates an interesting difficulty that could undermine the activity or could be turned to advantage, depending on how it is addressed. This difficulty arises because the chances of Player 2 (even) winning a point is $^{41}/_{81}$ compared to $^{40}/_{81}$ for Player 1 (odd), as revealed by analysis of the sample space. This miniscule difference in likelihood implies that the game’s unfairness is unlikely to be convincingly evident when playing “first to ten points”.

The interest is in how the teachers implemented the activity in the classroom, and in what they allowed it to exemplify and what students might have learned from it. Irene preceded her use of the game by getting students to toss a coin 100 times and record the number of heads and tails, with pairs of students starting to play the spinner game as soon as they had completed their 100 tosses. This meant that some students had more time to engage with the game than others, and that some of the important teaching moments occurred for small groups of students rather than the whole class. Most students had played the game for a few minutes before Irene interrupted them for a discussion of the coin tossing results and then the spinner game. Her focus here was really on the coin tossing results, and time constraints limited the attention given to the spinner game. Nevertheless, some of its attributes were addressed. She asked the class if they thought it was a fair game. Discussion ensued, as students posed various ideas without any of them being completely resolved. For instance, there was a brief discussion about how the “structure” of the game needed to be fair, implying that fairness means that as long as the two players play by the rules of the game then they should have an equal chance of winning. Most of the arguments about fairness were associated with the number of odds and evens, both in terms of the individual numbers on the spinners (there are more odds than evens on each spinner) and in terms of the sums. One student neatly articulated the erroneous parity argument, that since “odd + odd = even and even + even = even but odd + even = odd, therefore Player 2 has two out of three chances to win”. Irene said she was not convinced about the “two out of three”, but she agreed the game was unfair. Irene then allowed one of the students to present his argument. At the start of the whole class discussion this student had indicated that he had not played the game at all but had “mathsed it” instead, and at that time Irene made a deliberate decision to delay the details of his contribution until the other students had had their say. He proceeded to explain that he had counted up all the possibilities, to get 38 for even and 35 for odd. Although this was actually incorrect Irene seemed to believe that he was right and continued by pointing out that this meant that “it’s not *terribly* weighted but it *is slightly* weighted to the evens”. Irene then asked the class if their results bore this out, and highlighted that although the game was biased toward Player 2 this did not mean that Player 2 would always win.

Greg spent a much longer time on the spinner game. The students played it at the end of the first of the two observed lessons, and during the course of their exploration of the game a few pairs came up with the parity argument, accompanied by the observation that there are more odd numbers on the spinners. That lesson concluded with an extensive discussion of whether or not the game was fair. Greg did not indicate whether or not he thought the students' suggestions were correct; he seemed to want to hear all the contributions. He later asked if any of the students had considered all the possible outcomes, and suggested that this would something they would look in the next lesson. In the post-lesson interview Greg told the researchers that the decision to explore sample space was made only during the first lesson while students were already working on the task. He also acknowledged that when he chose the activity he was not sure of all that it offered.

Greg then devoted nearly half of his second lesson to an exploration of the sample space. As reported in Chick and Baker (2005b) he tightly guided the students in recording all the outcomes and could not deal with alternative approaches. He asked the students to calculate the probabilities of particular outcomes, which was helpful in highlighting the value of enumerating the sample space, but detracted from the problem of ascertaining whether even or odd outcomes were more likely. Students eventually obtained the "40 odds and 41 evens" conclusion, at which point Greg stated that because the "evens" outcome was more likely the game was unfair. There was, however, no discussion of the narrowness of the margin.

It must be noted that in both classes the students did not—could not—play the game long enough for the unfairness to be genuinely evident in practice, yet most students claimed that the game was biased towards even. This may have occurred because the incorrect parity argument made them more aware of the even outcomes than the odd ones.

As suggested earlier, the spinner game provides the opportunity to examine sample space, likelihood, and fairness. Given the impact of time constraints on Irene's lesson, sample space was not covered well, although she believed that the student who had "mathsed it" had considered all the possibilities. This highlights a contrast between her knowledge of his capabilities and the details of the content with which he was engaged. On the other hand, her content knowledge was sufficient for her to recognise the significance of the small difference between the number of odd and even outcomes and its impact on fairness. Greg was much more thorough in his consideration of sample space, but also very directive. He seemed constrained by his content knowledge, having only one way to think of the sample space—via exhaustive enumeration—and was unable to recognise the possibility of an alternative approach in one of his students' erroneous suggestions.

Neither teacher seemed aware of all that the game afforded in advance of using it, as evidenced by the way it was used, although Greg recognised the scope for examining sample space part way through the first lesson. Both teachers were, however, able to bring out some of the concepts in their use of the game, with Irene having a good discussion of the meaning of fairness and the magnitude of the bias, and Greg illustrating sample space and the probability of certain outcomes.

An important observation needs to be made here. The teacher guide that was the source of the activity gave too little guidance about what it afforded and how to bring it out. Even if such guidance had been provided, there is also still the miniscule bias problem inherent in the game's structure that affects what the activity can afford. It is very difficult to convincingly make some of the points about sample space, likelihood, and fairness with the example as it stands. It can be done, but the activity probably needs to be supplemented with other examples that make some of the concepts more obvious (see, e.g., Baker &

Chick, 2007). This highlights the crucial question of how can teachers be helped to recognise what an example affords and then adapt it, if necessary, so that it *better* illustrates the concepts that it is intended to convey.

Area and Perimeter

The final case involves Clare, a Grade 6 teacher with five years' experience. She conducted two lessons focussing on area and perimeter simultaneously, having done work in the past on each separately. Part of her first lesson is presented here in detail, to highlight the way the actual implementation of an example in the classroom may develop in unanticipated ways and to indicate how important PCK is in dealing with this.

Clare began by reviewing the concept of area, where she emphasised that "Area measures the *space* inside a shape, so what that actually is, is the number of squares inside the shape". She then asked students to draw a rectangle with an area of 20cm^2 on grid paper and cut it out. Her choice of what might be called an open "reversed" task was appropriate given that the students had worked with area before, including the area formula for rectangles. Shortly after this instruction the following exchange took place between Clare and a student.

S: Can I do a square?
Clare: Is a square a rectangle? […] What's a rectangle? […] How do you get something to be a rectangle? What's the definition of a rectangle?
S: Two parallel lines
Clare: Two sets of parallel lines … and …
S: Four right angles.
Clare: So is that [points to square] a rectangle?
S: Yes.
Clare: Excellent. [Pause] But has that got an area of 20?
S: [Thinks] Er, no.
Clare: [Nods and winks]

It is not clear whether Clare's original choice of 20 was made with any awareness of geometrical implications, but the fluency with which Clare moved from area measurement to spatial issues—addressed with clear attention to geometrical properties—and back again required ready access to the PCK of both the measurement and spatial domains. She also exhibited effective use of questioning to elicit understanding from the student. Shortly after this she discussed rectangle properties with the class.

Clare then invited a student to bring his 4×5 cut-out rectangle to the front of the class, recorded it on an overhead transparency, and confirmed that its area was 20cm^2. She led a class discussion on how multiplying length × width is the same as counting squares and hence gives the area. Clare thus used the concrete example to highlight the link between the conceptual meaning of area and the procedural calculation. She did not stop there, however; in the following exchange it can be seen that Clare knew that students need to know that the area formula L×W only applies to certain shapes.

Clare: When [S1] said that's how you find the area of a shape, is he *completely* correct?
S2: That's what you do with a 2D shape.

Clare: Yes, for *this* kind of shape. […] What kind of shape would it *not* actually work for?
S3: Triangles. […]
S4: A circle.
Clare: [With further questioning, teases out that L×W only applies to rectangles.]

A student then suggested 2×10 as a second example of a rectangle with area $20cm^2$, at which point Clare confirmed that all the students had chosen either this one or the 4×5 case. When she asked for other possibilities the students suggested the original examples but oriented at 90°, together with 1×20, which had not been suggested earlier. With all the integer-sided rectangles on display Clare asked the students to look for a pattern in the examples found, which led into a discussion of factors of 20. She continued:

Clare: Are there any other numbers that are going to give an area of 20? [She paused, with an attitude of uncertainty. There was no response from the students at first.]
Clare: No? How do we know that there's not?
S: You could put 40 by 0.5.
Clare: Ah! You've gone into decimals. If we go into decimals we're going to have *heaps*, aren't we?

It appeared that she was targeting only whole numbers—and, as a consequence, some argument about exhausting the factors of 20—but she clearly understood the significance of the student's unexpected answer, and to what degree it would apply. The open scope of her questions allowed this extension to arise, even though it had not been her original intention; however, she made a decision not to pursue this aspect—even though it would have been a valuable use of the $20cm^2$ example—because she wanted to move on to different examples that would highlight other relationships. Instead she used the $20cm^2$ example to focus on the search for all factors of 20.

This exploration of the $20cm^2$ example took the first 15 minutes of the lesson. Clare then had students repeat the search for rectangles with area $16cm^2$. She used this example to highlight the process of systematically searching for factors, and to highlight the set inclusion property "a square is a rectangle". She recapped that they had been working on areas, and then reminded students about perimeter, how to work it out for rectangles, and that linear rather than square units are involved. She guided the class to work out the perimeters of the different $16cm^2$ rectangles they had found, and indicated that although shapes might have the same area they do not have to have the same perimeter. She revisited the $20cm^2$ examples they had, and calculated the perimeters to focus again on the variation in perimeter.

The final example/task for the lesson was for students to work in groups to find as many shapes—not just rectangles, but constrained by being made of contiguous squares—with an of area 12 cm^2 and determine the perimeters. She wrote "What is the relationship between area and perimeter?" on the board as a learning objective for this activity. She allowed students to explore the task for about five minutes, then interrupted their work to help them develop strategies to work systematically and instruct them to record the perimeters of each shape. About 20 minutes later, she held another class discussion that acknowledged that there were "heaps" of possible shapes, looked at one group's systematic work and discussed some symmetry implications, and then asked students to focus on finding a shape with the greatest perimeter and one with the smallest perimeter. The 90-minute lesson concluded with a ten-minute discussion of the students' results, which emphasised the use of linear units for perimeter, that shapes with small perimeters were more "compact", and that moving one of the squares on a shape without changing the number of joining edges will not change the perimeter.

Clare's conclusion re-emphasised the points of her lesson: that area and perimeter can have the same or different numerical values, that two shapes with the same area can have different perimeters, and that systematic work can help find all the possibilities in a problem. These learning outcomes were achieved through the use of just three examples that had been carefully chosen to illustrate these points.

Clare seemed to have a very clear idea about what she wanted her examples to achieve. They were effective as didactic objects for two reasons: Clare's careful choice of the examples themselves and then the way she facilitated conversations about them. It is not clear that there was a purposeful reason for considering rectangles of area $20cm^2$ first, followed by those of area $16cm^2$; in particular, it is uncertain that there was an intention to allow discussion of "squares are rectangles" in the second case after just focussing on non-square rectangles. However, whether it was an intended focus or an opportunity that arose fortuitously, Clare was able to address this geometric concept fluently, demonstrating her capacity to make connections across topics. The final extension considered shapes of area $12cm^2$. If only considering rectangles this would have been no more difficult than what students had already done—and potentially redundant—but because she wanted students to consider other shapes as well, it was appropriate to pick this "simpler" number. Interestingly, given the magnitude of the enumeration task, there is potential to debate whether 12 is, in fact, simple enough. One of the researchers observing the lesson at the time wondered if she had chosen wisely. As the lesson progressed, however, it was clear that although she wanted to address the issue of enumerating all possible shapes systematically, her main focus was still associated with area and perimeter, and the choice of 12 allowed enough variety of shapes to make it a non-trivial task to find those with the greatest and least perimeter.

There was an interesting decision point that arose in the lesson when a student gave the 40×0.5 rectangle example. It seemed that Clare's focus on factors influenced her decision to acknowledge this response, briefly recognise its implications, but then continue with whole number dimensions. It is not clear whether she weighed up (a) what concepts could have been developed if she had detoured with an exploration of non-integer dimensions, (b) how such a detour might have interfered with her goals for the lesson, and (c) whether or not all her students would have been capable of following the detour. Certainly such an exploration could have given more extreme perimeter values than the students obtained, but the importance of identifying factors of numbers might have been obscured.

The strength of Clare's PCK was evident as the lesson progressed, as well as in her responses to the questionnaire and interviews (see Baker & Chick, 2006). She appeared to have a deep understanding of concepts, the rich connections among them, and the links between concepts and procedures. Her conceptual fluency was evident in the ease with which she responded to unanticipated events in the classroom. In addition to specific content knowledge she advocated general mathematical principles, such as the need to work systematically, and to justify and explain results. Her knowledge of student thinking was evident in her identification of likely misconceptions, and in knowing how to ask questions and respond to students' difficulties. Finally, her choice of examples had appropriate cognitive demand for her students, led to conceptual understanding, and afforded exploration of a range of mathematical concepts.

Conclusions

For most of these cases, the teachers selected the example's structure and specific values prior to implementing it in the classroom, strongly influenced by their PCK and what affordances they thought the example offered. At times, though, teachers had to develop or respond to an example on the spot; but again their capacity to do so was affected by their PCK and their ability to construct or recognise examples with the affordances required. It is worth making some observations about the source of the examples and the PCK for some of these situations, in order to highlight the complexity associated with this critical issue.

- A teacher's current level of PCK can allow him/her to recognise a situation that could be turned into a useful example, as evident in the use of the Aero bar.
- A teacher's current level of PCK may allow him/her to devise a partly appropriate example, but deeper PCK would reveal that it has limitations. This occurred with the Smarties and with Frank's fraction subtraction example.
- Professional development (PD) can enhance PCK and a teacher's repertoire of examples. The fraction wall and the paper strip folding activities conducted by Meg had their origins in PD and reflected, in her paraphrased words, part of a change in her teaching style from a procedural focus to a conceptual one. That said, however, a teacher's implementation of an example demonstrated to him/her in PD may not always reflect the potential affordances identified by the PD designers. The omission of the sevenths and ninths was Meg's choice; most advocates of the fraction wall would include these examples.
- External sources of examples do not always indicate the affordances of the example and how to implement them. This was strikingly evident in the case of the spinner game. It cannot be assumed that teachers do not or should not need this support.
- A teacher's current level of PCK and his/her identification of affordances can develop in the process of implementing an example. This occurred for Greg as he used the spinner game. Moreover, he recognised this development as such.
- A teacher with rich PCK can devise examples that illustrate a range of concepts, can highlight connections among topics, and identify which are the central ideas and which are peripheral. This was evident in Clare's area and perimeter examples.

The complexity of mathematical concepts, together with the limited opportunities that teachers have to master all these concepts and their pedagogical implications before entering the classroom, highlight how difficult it is to ensure that teachers have the depth of PCK required to identify and draw out the affordances of an example. Recognising the ways in which "Compare ${}^{2}/_{5}$ and ${}^{1}/_{3}$" is different from "Compare ${}^{3}/_{7}$ and ${}^{5}/_{8}$" and the consequent implications for what might be learned, for instance, requires attention to a range of fraction issues followed by a decision about which aspects are regarded as more important for the day's teaching objectives.

These observations raise the question of how to prepare future teachers so that they develop adequate PCK and can successfully choose, use, and modify examples. Clearly there must be an endeavour to ensure that teachers have a deep conceptual understanding of mathematics, and rich PCK for its teaching. Given the centrality of examples to the teaching and learning process, however, time also needs to be spent applying this understanding to an investigation of examples and their pedagogical implications. We need to develop ways to help teachers identify more potential affordances in examples, to recognise an example's salient and non-salient features, and to ascertain the implications of any interrelationships that exist.

This suggests that teacher education and professional development opportunities must be more explicit about the issues associated with example use. In particular, the affordances of the examples used in teacher education and professional development should be identified and discussed, so that teachers learn to realise that an example has many potential affordances and to discriminate between the productive and the unproductive. There is a need to identify the dimensions of possible variation for an example, so that the impact of changes to the particular values and structure can be considered, and the significant and extraneous components of the example can be identified. This is essential if teachers are to learn how to change examples to make them conceptually harder or easier, to produce counterexamples, or to emphasise a different principle. Indeed, teachers and potential teachers need opportunities to engage with examples, to trial them, and to learn how to adapt them successfully to meet different needs. It would be valuable to have teachers contrast examples, attending to affordances and what varies between the examples (the earlier illustration of examining the ways in which "Compare $^2/_5$ and $^1/_3$" is a different example from "Compare $^3/_7$ and $^5/_8$" is a case in point). In all of this, there needs to be deeper discussion of the connections among mathematical topics and how an example illuminates these connections. Finally, there must be discussion of how to implement the examples in the classroom, so that the examples become successful didactic objects that illustrate the desired general principle. Without this, the opportunities for learning afforded by examples may go unfulfilled.

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References

Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy. Final report*. London: King's College.

Baker, M., & Chick, H. L. (2006). Pedagogical content knowledge for teaching primary mathematics: A case study of two teachers. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 60-67). Sydney: MERGA.

Baker, M., & Chick, H. L. (2007). Making the most of chance. *Australian Primary Mathematics Classroom, 12*(1), 8-13.

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*(3), 241-247.

Bills, L., Mason, J., Watson, A., & Zaslavsky, O. (2006). *RF02:* Exemplification: The use of examples in teaching and learning mathematics. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 125-154). Prague: PME.

Bryman, A. (2004). *Social research methods* (2nd ed.). Oxford: Oxford University Press.

Chick, H. L., & Baker, M. (2005a). Investigating teachers' responses to student misconceptions. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 249-256). Melbourne: PME.

Chick, H. L., & Baker, M. (2005b). Teaching elementary probability: Not leaving it to chance. In P. C. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 233-240). Sydney: MERGA.

Chick, H. L., Baker, M., Pham, T., & Cheng, H. (2006). Aspects of teachers' pedagogical content knowledge for decimals. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Prague: PME.

Chick, H. L., Pham, T., & Baker, M. (2006). Probing teachers' pedagogical content knowledge: Lessons from the case of the subtraction algorithm. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 139-146). Sydney: MERGA.

Feely, J. (2003). *Nelson maths for Victoria: Teacher's resource Year 5.* Melbourne: Thomson Nelson.

Gibson, J. J. (1977). The theory of affordances. In R. Shaw & J. Bransford, (Eds.), *Perceiving, acting and knowing: Toward an ecological psychology* (pp. 67-82). Hillsdale, NJ: Lawrence Erlbaum.

Hill, H. C., Rowan, R., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematical knowledge for teaching. *The Elementary School Journal, 105*(1), 11-30.

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education, 6*(3), 223-252.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Leinhardt, G., Putnam, R. T., Stein, M. K., & Baxter, J. (1991). Where subject knowledge matters. In J. Brophy (Ed.), *Advances in research on teaching: Teachers' knowledge of subject matter as it relates to their teaching practice* (Vol. 2, pp. 87-113). Greenwich, CT: Jai Press, Inc.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States.* Mahwah, NJ: Lawrence Erlbaum Associates.

Marks, R. (1990). Pedagogical content knowledge: From a mathematical case to a modified conception. *Journal of Teacher Education, 41*(3), 3-11.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Skemp, R. R. (1971). *The psychology of learning mathematics.* Harmondsworth, UK: Penguin.

Stacey, K., Helme, S., Archer, S., & Condon, C. (2001). The effect of epistemic fidelity and accessibility on teaching with physical materials: A comparison of two models for teaching decimal numeration. *Educational Studies in Mathematics, 47*(2), 199-221.

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling, and tool use in mathematics education* (pp. 191-212). Dordrecht, The Netherlands: Kluwer.

Watson, A. (2003). Opportunities to learn mathematics. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity* (Proceedings of the 26th annual conference of the Mathematics Education Research Group of Australasia, pp. 29-38). Sydney: MERGA.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples.* Mahwah, NJ: Lawrence Erlbaum Associates.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91-111.

Watson, J., Beswick, K., & Brown, N. (2006). Teachers' knowledge of their students as learners and how to intervene. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 551-558). Sydney: MERGA.

Zaslavsky, O., Harel, G., & Manaster, A. (2006). A teacher's treatment of examples as a reflection of her knowledge-base. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 457-464). Prague: PME.

Zazkis, R., & Chernoff, E. (2006). Cognitive conflict and its resolution via pivotal/bridging example. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 465-472). Prague: PME.

Appendix 1.

A Framework for Pedagogical Content Knowledge (after Chick, Baker, et al., 2006).

PCK Category	*Evident when the teacher …*
Clearly PCK	
Teaching Strategies	Discusses or uses general or specific strategies or approaches for teaching a mathematical concept or skill
Student Thinking	Discusses or addresses student ways of thinking about a concept, or recognises typical levels of understanding
Student Thinking - Misconceptions	Discusses or addresses student misconceptions about a concept
Cognitive Demands of Task	Identifies aspects of the task that affect its complexity
Appropriate and Detailed Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams)
Explanations	Explains a topic, concept or procedure
Knowledge of Examples	Uses an example that highlights a concept or procedure
Knowledge of Resources	Discusses/uses resources available to support teaching
Curriculum Knowledge	Discusses how topics fit into the curriculum
Purpose of Content Knowledge	Discusses reasons for content being included in the curriculum or how it might be used
Content Knowledge in a Pedagogical Context	
Profound Understanding of Fundamental Mathematics (PUFM)	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept
Mathematical Structure and Connections	Makes connections between concepts and topics, including interdependence of concepts
Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident)
Methods of Solution	Demonstrates a method for solving a mathematical problem
Pedagogical Knowledge in a Content Context	
Goals for Learning	Describes a goal for students' learning
Getting and Maintaining Student Focus	Discusses or uses strategies for engaging students
Classroom Techniques	Discusses or uses generic classroom practices