Proportional Reasoning: Student Knowledge and Teachers' Pedagogical Content Knowledge

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This report considers the responses of 1205 students to two chance and data problems involving proportional reasoning and the interventions suggested by 44 teachers for four typical incomplete or inappropriate student responses. Students' responses reflect the relative difficulty of the two items, whereas the teachers' suggested interventions display a wide range of pedagogical content knowledge, including aspects of content knowledge and knowledge of students as learners. Rubrics are provided for both students' and teachers' responses and analysis supports the view that teachers find it difficult to assist students without directly telling them the answer. Suggestions are made for assisting teachers to improve feedback for students.

Following the work of Shulman in the 1980s (e.g., 1987) outlining seven types of teacher knowledge for successful teaching, researchers have focussed on various of the seven and combinations of them (e.g., Kanes & Nisbet, 1996; Mayer & Marland, 1997). Watson (2001) developed a profiling instrument for teachers of chance and data attempting to include all seven of Shulman's types of knowledge, whereas Hill, Rowan, and Ball (2005) appeared to incorporate content knowledge, pedagogical content knowledge, and knowledge of students as learners, into a rich description of "teachers' knowledge for teaching mathematics." This construct was assessed with a series of complex multiple choice items. Chick (2007) included these three types of knowledge, as well as curriculum knowledge, in her expanded framework for pedagogical content knowledge. It appears there is general recognition in the research community of the complexity of the interaction of the types of knowledge leading to teachers' successful implementation of learning programs in mathematics for their students.

Watson, Beswick, and Brown (2006) focussed on teacher content knowledge and knowledge of students as learners in asking teachers to suggest appropriate and inappropriate solutions that would be given by their students to a fraction problem. Further, pedagogical content knowledge was explored in asking teachers how they would use this problem in the classroom and intervene to address the inappropriate responses. Three tasks (8 individual items) of this type were included in the profiling instrument used in the current study but as well four new items were developed based on two items from student surveys. For these items actual student responses were presented to the teachers, rather than asking the teachers to suggest inappropriate student responses. Teachers were then asked how they would respond to the students' answers. A subscale of the teacher profile consisting of the 12 items relating to teachers' responses in the context of students' answers was considered by Watson, Callingham, and Donne (in press) as a measure of the wider interpretation of pedagogical content knowledge (PCK), including content knowledge and knowledge of students as learners. Watson et al. used Rasch (1980) measurement approaches and identified a single dimension of PCK, with few teachers reaching the higher ability levels of the scale.

Proportional reasoning is well-known for causing difficulty for many middle school students (Lamon, 2007; Mitchelmore, White, & McMaster, 2007). It is a necessary prerequisite for performing at the highest level of statistical literacy understanding (Watson & Callingham, 2003) and hence a topic of interest for exploring teachers' PCK in this context. Having data on both student performance and teachers' suggested interventions should provide starting points for professional learning within the larger project in which this study lies. The research questions for the study are hence the following.

What are the distributions of understanding of proportional reasoning shown by students in two chance and data contexts and the association between them?

What are the initial levels of PCK shown by middle school teachers in relation to remediating students' inappropriate responses to proportional reasoning questions in two chance and data contexts?

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Methodology

Sample. The sample consisted of 1205 students in grades 5 to 10 across 19 schools in the states of South Australia, Tasmania, and Victoria, whose teachers were involved in the StatSmart project (Callingham & Watson, 2007). Table 1 shows the distribution of the students across grades. The students were from the classes of the 44 teachers who completed the profiling survey. The teachers represented government, Catholic, and independent schools; 23 were male and 21 were female. The highest grades taught by the teachers were primary by 8, middle by 5, junior secondary by 9, and senior secondary by 21 (one unknown), but all teachers taught a class of students in one of grades 5 to 10 as part of the StatSmart project. The years of teaching experience ranged from 1 year to more than 25 years. There were 12 teachers in the longest serving group, with 12 others having taught 16 to 25 years or 6 to 15 years, and 8 having taught 5 or fewer years.

Table 1

Number of Students by Grade

Grade	5	6	7	8	9	10	Total
Number of students	41	175	432	324	137	96	1205

Items. The two problems completed by the students are shown in Figure 1. The top problem, named BOX, was originally adapted from an item used by Konold and Garfield (1992) and analysed for Australian students by Watson, Collis, and Moritz (1997) and Watson and Moritz (1998). It was part of the survey used to define the hierarchy of statistical literacy by Watson and Callingham (2003). The lower item, named SMOKE, was an item employed by Batanero, Estepa, Godino, and Green (1996) and further analysed by Watson and Kelly (2006). Watson and Callingham (2005) used the item as part of a confirmatory study of the statistical literacy construct.

For each of these two problems answered by students, two student incomplete or inappropriate answers were presented to teachers. Items T1 and T2 in Figure 2 are associated with the BOX question and items T3 and T4 are related to SMOKE. The original student questions were in the teacher profile but are not repeated here. The student answers were chosen as typical responses from students in the previous studies.

Box A and Box B are filled with red and blue marbles as follows. Each box is shaken. You want to get a blue marble, but you are only allowed to pick out one marble without looking. Which box should you choose?

Box A	Box B
6 red 4 blue	60 red 40 blue

(A) Box A (with 6 red and 4 blue).

(B) Box B (with 60 red and 40 blue).

(=) It doesn't matter.

Please explain your answer.

The following information is from a survey about smoking and lung disease among 250 people.

	Lung disease	**No lung disease**	**Total**
Smoking	90	60	150
No smoking	60	40	100
Total	150	100	250

Using this information, do you think that for this sample of people lung disease depended on smoking? Explain your answer.

Figure 1. Two proportional reasoning problems used in student surveys.

	Consider each of the following answers and explanations to the problem and discuss how you would respond to the answers.	
BOX	T1 Student 1: (=) Because you could get red or blue.	T2 Student 2: (A) Because there are only 2 more reds in A and 20 more in B.
SMOKE	T3 Student 3: Yes, 90 who smoked got lung disease.	T4 Student 4: [No] 60 "no smoking lung disease" and 60 "smoking no lung disease" are the same.

Figure 2. Teacher items based on proportional reasoning problems (Figure 1).

Analysis. The rubrics for the student responses were those used in previous research. For the BOX item a four-step rubric as used by Watson and Callingham (2003) is given in Table 2, consolidated from the seven-step rubric used by Watson et al. (1997).

Table 2

Rubric for BOX Item for Students

Code	**0**	**1**	**2**	**3**
Description	No response/reason	Anything can happen	Additive reasoning	Multiplicative/ Proportional reasoning

For the SMOKE item a five-step rubric was devised from the one used by Watson and Callingham (2005), based on a developmental model of considering increasing numbers of elements in the two-way table and combining them in an appropriate fashion. The rubric is summarised in Table 3. The difficulties of the different item-steps were obtained from Rasch measurement so that the relative difficulty of the two items could be considered.

For items presented to teachers, the student answers (see Figure 2) were coded as: BOX items, Code 1 for T1 and Code 2 for T2; SMOKE items, Code 2 for T3 and Code 3 for T4. The rubric in Table 4 was used to code each of the teacher items. The raw scores for students and teachers on each rubric were used as a basis for the analysis. Student items were considered by grade. The association between the two items for both teachers and students was considered using two-way tables.

Table 3

Rubric for SMOKE Item for Students

Code	**0**	**1**	**2**	**3**	**4**
Description	Yes/No without justification; No reason/ response	Yes/No no use of data; knowledge of context	Yes/No single comment on method or single cell data	Yes/No explicit or implicit use of 2 or 3 cells' data	No uses all information with ratio/ percents

Table 4

Rubric for Teacher Responses to T1 to T4

Code	0	1	2	3
Description	No/irrelevant response	General strategy; no math. input	Some math. content but vague teaching strategy	Questioning of students; multiple aspects of problem; prop. reasoning strategies; cog. conflict

Results

Research Question 1 – Student Outcomes. The Rasch output showed that SMOKE was, in general, more difficult than BOX for students. Figure 3 shows the relative difficulty in logits of each item-step. The difficulty level of BOX3 was about the same as that of SMOKE2 and a little lower than SMOKE3. It appears that proportional reasoning was more difficult to achieve in the context of SMOKE than BOX. This finding seems consistent with what might be expected, based on the general complexity of presentation of the two items. The jump from BOX2 to BOX3 and for SMOKE3 to SMOKE4 suggests that, regardless of context, it is difficult for students to move to proportional reasoning, relying on relationships between the numbers presented.

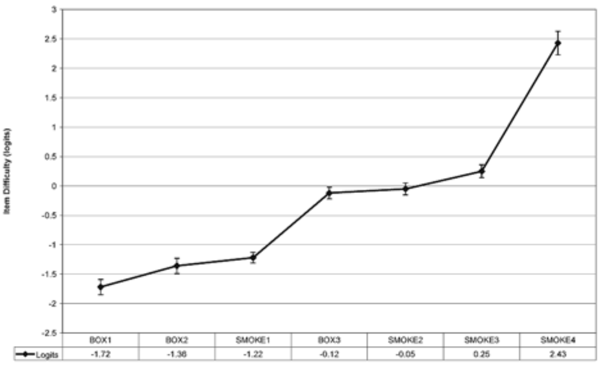

Figure 3. Relative difficulties of item-steps for BOX and SMOKE.

Table 5 contains a summary of the outcomes by grade for each code of the two items and Table 6 contains typical student responses for each code of each item from this data set. For the BOX item, except for grade 9, which was slightly lower and grade 6, which was slightly higher, the frequency of no response or no reason was relatively uniform across grades. Only grade 6 seemed susceptible to "anything can happen" reasoning. Improvement was seen across the pairs of grades 5/6, 7/8, and 9/10 for the appropriate reasoning. Not until grade 9/10 was there a moderate improvement in the appropriate reasoning for the SMOKE item.

Table 5

Percent at each Code by Grade for the Student Items

Code	Gr5 (n=41)	Gr6 (n=175)	Gr7 (n=432)	Gr8 (n=324)	Gr9 (n=137)	Gr10 (n=96)	Total (n=1205)
BOX0	12.1	17.7	9.7	12.3	5.8	10.4	11.3
BOX1	2.4	16.0	5.8	8.6	0.7	2.1	7.1
BOX2	58.5	40.0	39.4	37.3	27.0	25.0	37.0
BOX3	26.8	26.3	45.1	41.7	66.4	62.5	44.6
SMOKE0	29.3	28.0	26.4	18.8	24.1	20.8	24.0
SMOKE1	31.7	48.6	33.6	33.3	21.9	24.0	33.5
SMOKE2	7.3	6.3	10.4	11.4	16.1	9.4	10.5
SMOKE3	29.3	15.4	27.1	33.0	28.5	40.6	28.3
SMOKE4	2.4	1.7	2.5	3.4	9.5	5.2	3.7

Table 6

Examples of Student Responses at each Code Level for the Student Items

Code	Response
BOX0	"A"
BOX1	"=, because it's just a luck you get"
	"=, because it would be a 50-50 pick"
BOX2	"Box A because there is only 2 more red where as Box B has 20 more red"
	"=, they both have more red than blue"
BOX3	"=, because there is an equal chance that you will get a red or blue marble"
	"=, because 60% of the marbles in each box are red"
SMOKE0	"Well I don't really understand because you don't really know"
	"Yes because there are more people"
SMOKE1	"No, smoking is what gave them the disease in the first place"
	"No because it could have been a bushfire or second hand smoke"
SMOKE2	"Yes, because there is 90 people that got it that do smoke"
	"No, not really they need to do the survey on an even amount of people"
SMOKE3	"Yes, there are 60 people with lung disease not smoking and 60 people smoking and no lung disease"
	"Yes, because the people that do smoke had more people getting lung disease than people who don't smoke"
SMOKE4	"No they both had the ratio of 3:2 (smoking : non smoking)"
	"No, because 2/3 of non-smokers have lung disease as well smokers having 2/3"

Table 7 shows the association of the code levels for the two items, again illustrating their comparative difficulty. Achieving Code 3 on the BOX item was no guarantee of even moderate achievement on the SMOKE item, as 107/538 (19.9%) received a Code 0 and only 36/538 (6.7%) received a Code 4. Of those who received a Code 4 on the SMOKE item, 36/44 (81.8%) received a Code 3 on the BOX item.

Table 7

Association of Code Levels for BOX and SMOKE

		BOX				
	Code	**0**	**1**	**2**	**3**	**Total**
SMOKE	**0**	58	14	110	107	289
	1	44	52	159	149	404
	2	5	4	47	71	127
	3	26	15	125	175	341
	4	3	0	5	36	44
	Total	**136**	**85**	**446**	**538**	**1205**

Research Question 2 – Teacher Outcomes. Table 8 shows the percentage of teachers achieving each code for the four items in Figure 2. As can be seen the modal response for all four items was code 2, reflecting limited engagement with the mathematics and potential teaching strategies. The difficulty of the item for students hence did not appear to influence the code achieved by the teachers. Examples of teacher responses at each code level are given in Table 9.

Table 8

Percent of Teacher Responses at each Code Level for each Item

	Teacher survey questions			
	T1(BOX)	**T2(BOX)**	**T3(SMOKE)**	**T4(SMOKE)**
Code 0	20.4% (9)	11.4% (5)	9.1% (4)	20.4% (9)
Code 1	25.0% (11)	9.1% (4)	13.6% (6)	22.7% (10)
Code 2	43.2% (19)	61.3% (27)	56.8% (25)	47.7% (21)
Code 3	11.4% (5)	18.2% (8)	20.4% (9)	9.1% (4)

In considering the association of teachers' responses between items, the strong representation of Code 2 across all items meant that Code 2 was frequently common across pairs of items, ranging from 27.3% to 38.6% for the six pairs (e.g., T1 & T2, T1 & T3, etc.).

Discussion

Several aspects of the results suggest implications for teachers and teaching related to proportional reasoning. These include the relative difficulty of the items for students, the lack of discrimination of teachers over the four student responses considered, and suggestions for improving teacher interaction with student responses.

Although the literature agrees that proportional reasoning is difficult for middle school students, the outcomes of this study suggests that ratios that are multiples of ten are much easier to recognise than those with a non-integer multiple, that is, such as 1.5 in the SMOKE problem. As well the presence of row and column totals in the SMOKE table more than doubles the amount of information that needs to be taken in, in engaging with the question. The cognitive load certainly is greater. For most students there appear to be few similarities in the visual presentation of the problems.

The lack of variability of the level of teachers' responses across the student answers of different codes would appear to reflect a general lack of PCK at the point of matching content knowledge with knowledge of students as learners. Knowing what questions to ask of students or what cognitive conflict to generate, without directly telling them the answer, appears to be a difficulty for these teachers. Teachers are presenting the same types of generic responses regardless of the level of student response and not recognising an appropriate zone of proximal development in which to challenge the students. This may also be influenced by a lack of appreciation for the desired level of response expected in the profiling instrument. It is hoped that professional learning during the StatSmart project will increase teachers' familiarity with their students' understanding and ways of matching that to the appropriate mathematical content. Specific analysis of the levels of student response within a structural model should also assist in increasing the teachers' PCK.

Table 9

Examples of Teachers' Responses at each Code Level for each Item

Code	T1(BOX)	T2(BOX)	T3(SMOKE)	T4(SMOKE)
0		"Correct, but which colour would you more likely get?"	"unsure"	"unsure"
1	"I'd get them to test this not just think about it." "Not sure what question I would ask – need to look at ratios but not sure how to go about it?"	"What chance do you have? More chance of getting either?" "Can you tell me more? This student needs to explain more fully how he arrived at his answer. He may or may not be correct; only a fuller explanation could tell."	"Of what number in total? Ask questions which give meaning to the numbers."	"We would look more closely at the chart and be more accurate with analysing the data under each heading." "I would change the numbers so that doesn't add to the confusion. Then discuss the categories and what they mean."
2	"This student does not understand fractions or proportion Red 6/10 or 60/100 = This student needs more work with concrete aids." "In both cases I would talk about probability and the likelihood of choosing either or percentage and ratio of either blue or red in each box."	"Talk about chance of getting blue over red or vice versa. I'd get them to try both and test their theory." "But there are more blue in Box B?" "In both cases I would talk about probability and the likelihood of choosing either or percentage and ratio of either blue or red in each box."	"Need them to look at 90 out of 150 versus 60 out of 100 – probably look at percentages and ratios." "We would look at the total amount of people being surveyed in each section and try to convert the numbers so we could compare the answers more easily."	"Not a fair answer as totals are different e.g. smokers, 60 out of 150." "May look at ratios – would need to make non smokers = 150 to show clearly."
3	"Are there also more blues in B? If I have less marbles do I have a better chance always?"	"Question: Is it all luck? Would I be more likely to get a red? Why? In both boxes? How could I show my chance of getting a blue in numbers?"	"From which sample? What % of smokers got the disease? What % of no smoking got the disease? Same!! 90/150 = 3/5 = 60% (smokers) and 60/100 (ns)" "I would encourage student to read the whole table and look at all the data. Do high numbers mean a greater chance?"	"I would ask the student to look carefully at the size of the sample groups for smoking & non-smoking and compare the number of %'s of people with & without lung disease. Have they recognised that the sample groups are different sizes, what does this mean when looking at the results?"

The suggestions for professional learning include making teachers aware of the three stages of introducing mathematical concepts in context: terminology, terminology in context, and critical thinking (Watson, 2006). In this case, the terminology associated with proportional reasoning needs to be understood. Then it needs to be related to contexts such as chance (the BOX problem) and data (the SMOKE problem), with the differentiation pointed out. Finally critical thinking to answer questions that create conflict between mathematics and other contextual understanding (e.g., beliefs about smoking and lung disease) needs to be made explicit. In conjunction with this framework, the suggestions of Chick (2007) in terms of the affordances of using examples in the classroom, are also likely to be beneficial.

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