Being Mathematical, Holding Mathematics: Further Steps in Mathematical Knowledge for Teaching

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Discussion, theorising and research in the area of mathematical knowledge for teaching began by asking what level of mathematical qualification teachers need. We have moved on considerably since then, using concepts such as pedagogical content knowledge, and frameworks such as those from Deborah Ball in Michigan or, more recently, Tim Rowlands in Cambridge or Alan Schoenfeld in Berkeley.

Some lovely research looking at teachers in action and asking questions about what they bring to their mathematical teaching decisions has pushed our ability to adequately describe the complex environment of the teacher. We are left with the feeling that the practice is still not properly understood.

Anne Watson in Oxford has recently written about "mathematical modes of enquiry"--mathematical practices that teachers emulate in ways appropriate to the classroom. I will discuss this concept through examples, and propose an extension to it that looks not just at what we know mathematically, nor how we behave mathematically, but also at the way we know: how we hold mathematics.

I want to talk about mathematical knowledge for teaching. My own work is mainly with secondary and undergraduate teaching, but I am increasingly discovering that the main issues transcend the levels of mathematics. You, the reader, will probably need to adapt any ideas of interest to your own classrooms.

I believe that, so far, we have not properly understood some important dimensions of Mathematical Knowledge for Teaching (MKfT). The evidence is in classroom research—we are far from capturing what it is a teacher does, why they do it, and what effect it might have on student learning.

We do not yet have theories (let alone research evidence) that can explain why some highly mathematically qualified people who are highly motivated to be teachers are unsuccessful—and why some mathematically unqualified teachers produce students who achieve at the highest levels. Nor can we explain the wide variation in achievement by students who have experienced the same classroom environment. Nor have we come close to linking specific mathematical learning with specific teacher behaviour.

The Development of MKfT Research & Theorising

"How much mathematics should a teacher study?" This, and similar research questions, were never going to be properly answered, but a quick web search found attempts to correlate mathematical qualifications (and other teacher characteristics) since 1972 up to the present (Begle, 1972; Buddin & Zamarroa, 2009). Contradictory results abound, although one is left with the feeling that, particularly at lower grades, mathematical qualifications are loosely connected to student achievement. From Yr 7 upwards at least, my own rule of thumb is that a teacher needs mastery of mathematics a minimum of four years above the level at which they teach (Paterson, Horring, & Barton, 1999). But even providing an answer to the original question does not get us very far unless we are only interested in pre-service teacher education policy-making.

I guess that it was partly dissatisfaction with this type of research question that led Schulman to propose the three categories of general pedagogical knowledge, general knowledge of subject matter, and pedagogical content knowledge (Schulman, 1987). The insight that not only did a teacher need pedagogical knowledge and content knowledge, but

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that these two interacted into a special form of knowing has kept researchers busy for many decades. The phrase "pedagogical content knowledge" includes knowledge about how mathematical topics are learned, how mathematics might best be sequenced for learning, having a resource of examples for different situations, and understanding of where conceptual blockages frequently occur, and knowing what misunderstandings are likely. Questions remain about how teachers best come by this knowledge, the extent to which it can be taught and the extent to which it depends on experience, and, inevitably, the hard question: what is the relation of this type of knowledge to student learning?

In this paper I wish to focus, however, on mathematical knowledge for teaching—not on pedagogical knowledge, nor on pedagogical content knowledge. Let me note two things. First, I unequivocally regard these as vital components of knowledge for teaching. They are just not my current subject.

Second, I believe that we ignore the mathematical part of teachers' knowledge at our peril. When I consider most accepted advances in teaching practice (things like communicative teaching and learning, constructivism, collaborative learning, integration of technology), then I realise that each of these requires more mathematical understanding on the part of the teacher than ever before. The demands in pure content knowledge keep increasing, the need to understand the subject in a wider sense keep increasing, the list of applications and careers that students are heading towards keep increasing. Such a realisation is at once both frightening and exciting.

To return to theorising about MKfT. Several theoretical models of knowledge for teaching have emerged, and all have a place in them for mathematical knowledge. All move on from a simple list of syllabus topics—they include areas like the history of mathematics and applications of mathematics. Deborah Ball's team at Michigan University watched primary teachers in action, and using these observations developed the following model, or map, of the knowledge they saw being used by teachers (Hill, Blunk, Charalambos, Lewis, Phelps, Sleep, & Ball, 2008). On the Content knowledge side of their model are three sections: specialised content knowledge, common content knowledge, and knowledge at the mathematical horizon. This latter implies forward knowledge: not the mathematics that is being taught but that which is being prepared for.

Tim Rowland and his team at Cambridge University similarly developed a model based on observations of teachers. Their Knowledge Quartet has the dimensions foundation (for example, overt subject knowledge), transformation (for example, choice of examples), connection (for example, decisions about sequencing), and contingency (for example, responding to children's ideas), describing what teachers do with their mathematical knowledge (Rowland, Turner, Thwaites, & Huckstep, 2009, p.28-33).

A third example of a model of MKfT is the work of Alan Schoenfeld at Berkeley (Schoenfeld, 2002). He approached the subject from a theoretical point of view and then applied his theoretical framework to videos of actual teaching. The KOG framework (Knowledge, Orientations, Goals) enables him to explain teacher decisions in the classroom from the inferred KOG of the teacher. He argues that if one knows sufficiently well the knowledge, orientation and goals a teacher has in a classroom situation, then this is sufficient to explain all teaching decisions.

All of these models focus on WHAT the teacher must know, but bring in the link with pedagogy and/or wider knowledge or attributes of the teacher. I wish to turn our attention to HOW a teacher must know.

Being Mathematical

Anne Watson, from Oxford University, talks about Modes of Mathematical Enquiry. She believes that this a key component of MKfT (Watson, 2008).

What are modes of mathematical enquiry? These are the things that mathematicians do. Examples include conjecturing, generalising and abstracting, persisting, making links, arguing, justifying, and proving.

Watson does not mean that teachers must know what these modes of mathematical enquiry are so that they can teach them to their students. No, she means that teachers must embody modes of mathematical enquiry themselves—they must do these things in the classroom, in appropriate ways. Teachers must be mathematicians. She believes, and I agree, that this can happen appropriately at any level.

The reason for a teacher to be a mathematician is not just so that they model the behaviour that they wish a student to adopt. It is more than that. In order to learn mathematics, students must be in a mathematical environment, they, too, must become mathematicians in some sense, not just learn about mathematics from the outside. Therefore the teacher must know how to make a classroom a place in which mathematics is performed. It must be a mathematical environment.

Abstraction

Let us take the mathematical behaviour of abstraction. What would this look like in a mathematics classroom?

(I'm going to take a deep breath and plunge out of my depth—into a primary environment. Please forgive my naivety in some of my contexts but take the idea and see whether it works).

A first example: I imagine that, in many primary mathematics classrooms teachers and students play with number sequences. Whether done through board games and dice, or formally by each student thinking up a "rule" for a sequence and trying to work out each other's secret. These are (in my view) valuable mathematical experiences.

By the way, the playing sequencing board games are not pre-mathematical. Keith Devlin, (2001) says that we all have the ability to do mathematics, because we all have the ability to gossip. In mathematics we do not gossip about our neighbours or our workmates. We gossip about abstractions. We discuss their relationships, their characteristics, their behaviour. If this is true, then what children need most, especially at a young age, is to have a lot of mathematical experiences of abstraction in its many guises. Luckily, children are very, very good at playing—much better than us. We, as teachers, need to re-learn how to play in order to give our children the experiences they will build on.

An example. My granddaughter Veronica and I are jumping down some steps. We count in turn 1, 2, 3, 4, … Then we start again and I count: 2. Pause. "What happened to 1?" "I don't like 1". Silence. We jump another step, and I say "4". "What happened to 3?" "I don't like three". Silence. And another "6". "What happened to five?" but now it is a rhetorical question. She knows the rules, and is soon counting with me. Numbers are no longer cardinal. They have become a sequence that can have its own rules.

So, lots of abstraction experiences. So far so good. But what is the mathematical experience of abstraction? It does not stop at one level. The whole power of mathematical abstraction is to then make the abstractions objects themselves, and available for further abstraction. For example, the idea of permutations (123, 132, 213, 231, 312, 321) led to the concept of a group, and definitions of groups and group-like objects, operations on groups, and so forth. Groups were identified in many other contexts (for example, symmetry). Groups were subsequently abstracted themselves and become one kind of object in Category theory.

What would an abstraction of an abstraction look like with the sequencing games of young students? Perhaps teachers do this, but what I would hope is that many many sequences are generated. As a result the sequences are open to being first objectified (given names will do, I do not mean write out formal rules), and then used as objects: combined, classified, compared, operated on. How do you “add” Veronica’s “two-step” with Cass’ “three-step”? What does it mean to “double” a sequence?

Could we then hope that sequence classifications might emerge and be named? Note that I am NOT hoping that children would see the difference between additive and multiplicative sequences (heavens no: arithmetic and geometric sequences are not part of the curriculum until Yr 12, talking about them in Yr 5 will not do!). I hope (and expect) that the children will dream up sequences that I have never heard of, will think of random sequences, Fibonnaci-type sequences, sequences modulo some number, backwards sequences, all sorts—and they will think up their own way of classifying (does the sequence contain my age; how many steps to get past 100). The point of this exercise is not to move the children faster into (or even give them a preview of) conventional mathematical topics—it is to give them experiences of abstraction and to show them what this really means. Having made classifications, might children start operating on them?

Here is a second example in the field of Geometry. What is an abstraction in geometry? Well, circles, triangles and other defined shapes are themselves abstractions (or idealisations) of the shapes we see around us. What about abstractions of position? Our children play with many of them: maps, networks, coordinates. These are familiar and I am confident that children get plenty of experiences of such abstractions: there are many imaginative teachers and creative resources in the area.

But again, that is only the first step. What can we do to encourage students to make abstractions of abstractions? Consider asking children to draw a map or plan of their house. But what sorts of maps are possible? What about a map showing the areas “belonging” to various family members, a map showing the house by temperature, a map showing the house in terms of time spent in each room by the child. How can we compare these maps? How can we merge maps of the same house? What about merging maps of the same kind but of different houses? This last would lead us to abstractions of “house”.

Teachers and mathematics educators, in my opinion, need to know much more about abstraction. We need to be able to do it ourselves in many situations, we need to make it happen mathematically in our classroom, we need to research how abstraction develops in students. I believe that when students have difficulty in mathematics we make a mistake if we go back to concrete examples. This will not help (we know it does not help). We need to go back to abstracting experiences that build towards mathematical abstractions. (Yes, it does require sufficient concrete examples to be in the student’s repertoire—but not as exemplars of what they are supposed to do, only as resources for them to form abstractions).

A consideration of abstraction leads us to an important reason why a teacher must know mathematics well in advance of the level they teach—because they are preparing the ground and the experiences for the abstractions that are yet to occur. They must teach mathematics not because this bit of knowledge is important, but because these experiences will allow a student to build future mathematical concepts.

Abstraction is just one of Watson’s mathematical modes of enquiry. We could do—I invite you to do—a similar analysis of generalisation, argumentation, linking, or creating mathematics is possible at any level of education.

Holding Mathematics

So far the discussion has been about knowledge in the sense of “knowing that” (I know that the sum of the interior angles of any triangle in a plane is π radians), “knowing about” (I know about the way the Greeks developed the deductive method through geometry), “knowing how” (I know how to solve a quadratic equation), and “knowing to” (I know to make conjectures and test them out). But there is still something missing in our description of the mathematical knowledge of a teacher—it is something broader and more fundamental. I describe it as “knowing mathematics” in the sense that you might say “I know my brother”: I know his moods, what makes him tick, why he behaves in certain ways, where his strengths and weaknesses are, where he came from and what he can and cannot do.

Hence my contribution to the discussion of MKfT is to talk about how a teacher holds their mathematics. Mathematical knowledge for teaching is more than the mathematics courses a teacher has mastered, it is more than the associated knowledge they have about applications or history, and it is more than knowing how to be mathematical. I suggest that a key factor in transforming these aspects of mathematical knowledge into effective teaching lies in a teacher’s attitudes and orientations towards mathematics: the way they hold their mathematics, the way they know mathematics, their relationship with mathematics.

Let me try to describe this aspect of mathematical knowledge more clearly. It may help to consider the following four (interrelated) components: a teacher’s vision of mathematics; a teacher’s philosophy of mathematics; a teacher’s sense of the roles of mathematics in society, and a teacher’s orientation towards the subject. (Let us refer to these as a teachers’ VPRO—Vision, Philosophy, Role for mathematics, and Orientation).

By a vision of mathematics I mean their concept of the discipline as a whole: what is the content of mathematics, how does it develop, what does mathematics deal with, what can it do and not do?

By a philosophy of mathematics I mean the way a teacher will answer questions such as: what is mathematical knowledge like, where does mathematics come from, how do we come to know mathematics, what is the status of mathematical objects and mathematical truths?

By a sense of the place of mathematics I mean how a teacher relates mathematics to other areas of knowledge, how the consider mathematics can affect society for better or worse, and what a teacher thinks mathematics is for—why should we value mathematical activity?

By an orientation to the subject I mean the way a teacher personally approaches mathematics: do they engage with it as a creative exercise, as a formal structure, as a set of facts, as a solitary activity, or as a game to be played with others? Do they regard it as awesome, as beautiful, as having infinite potential—or perhaps as formal, as a bare structure, as pure reason?

Felix Klein, one hundred years ago, wrote a book called Elementary Mathematics from an Advanced Standpoint (2004). As someone who might claim to have an understanding of the whole field of mathematics (a claim not possible today), he was challenging the senior secondary teachers of his day to present to their students a vision of the discipline of mathematics. The vision he presented was of a field that was connected, organic, and relevant. He believed that mathematics was a growing network, that its history was important to an understanding of the subject, and that its applications were at the heart of the nature of mathematics. Klein’s work is a famous example of how one can hold mathematics.

The way a teacher holds their mathematics is, I think, very deep knowledge, very personal knowledge, and it is not knowledge that we have worried about in the past.

What is interesting is that it does not seem to matter too much WHAT vision a teacher has, so long as they communicate it to their students. It does not seem to matter too much WHAT philosophy of mathematics a teacher has, so long as they have one and can argue it. It does not seem to matter too much WHAT role they believe mathematics plays in society, so long as they do have a role for it and can articulate it strongly. It does not seem to matter too much WHAT orientation they have (formal, playful, creative), so long as they have one and exhibit it.

The point is that a teacher demonstrates the way they hold their knowledge: they communicate their vision, they an argue for their philosophy, they articulate their place for mathematics, they exhibit their orientation to the subject. Hence the more substantive their VPRO, the more it will be part of their teaching.

Let us turn, for a moment, to the students. My argument is that students, as an important part of learning mathematics, need to develop their own VPRO for mathematics. They need to do this just as much as they need to learn mathematical facts, techniques, and modes of behaviour. If a teacher is not displaying some VPRO towards the subject the student will, in turn, not know about these key attributes to their learning and will not be able to develop their own VPRO through considering another's and interacting with it—agreeing, disagreeing, discussing, arguing, trying out different views, building towards a personal hold on mathematics.

Research

Just as there have been empirical studies that attempted to correlate mathematical qualifications with mathematical teaching ability or student learning; just as there have been analyses of classroom teaching that have mapped different sorts of mathematical knowing in relation to pedagogy; just as there are now studies being undertaken to observe mathematical modes of enquiry in mathematics classrooms; so too we also need research to investigate the nature and effect of the way a teacher holds their mathematics.

What will such research look like? First I suggest we need to learn to identify for ourselves different ways of holding mathematics, to observe the way a different vision or philosophy of mathematics is displayed for students, to see how mathematics' roles in society are communicated and how a teacher's orientation to the subject emerges in their classroom practices.

Then we need to work with students to understand whether they sense these differences, and what effects it has on their own holding of mathematics. Eventually we need to have some way of knowing how this affects their learning of the subject.

I will make two hypotheses about this kind of knowledge for teaching. I suggest there are two characteristics of the way a teacher holds their mathematics that are helpful for teaching and learning. They are the characteristics of openness and contingency.

I hypothesise that we will see that, for effective learning to take place, a teacher needs to be open to other ideas and the ideas of others. That is, while it is important for a teacher to have and display rich VPROs, it is necessary that they also admit the VPROs of others. More than that, a teacher ought to know some alternative VPROs to their own, so they can be recognised and brought out for examination. That is, I suggest we will find that it is helpful if teachers have some understanding of alternatives to their own views, but at least they must be open to the diverse, developing views of their students.

A second hypothesis is that we will find better teachers hold their mathematics contingently. It was suggested above that it does not matter what VPROs a teacher holds, so long as they hold and communicate them. But I believe that is not quite right. It does not matter what VPROs they hold but it not helpful for their students' learning if they hold these VPROs in a rigid manner. The reason I believe that we will find contingent holding of mathematics to be related to better teaching is not just because philosophies of mathematics have been developed over thousands of years and thus there is no basis for claiming that a philosophy to be the right one, the only true one, a philosophy that will not change. Nor is it because the roles of mathematics in society are different in different societies, and also change and develop. But rather, the connection with good teaching will be because a teacher who holds their VPRO rigidly cannot possibly be an active learner of mathematics.

Conclusion

I have tried to move forward our thinking about Mathematical Knowledge for Teaching.

From thinking about topics that mathematics teachers must know, I have moved through wider aspects of mathematical knowledge, through acting like a mathematician and creating a mathematical environment, to how a teacher holds mathematics.

Although a vision of mathematics, ideas of philospy, the relation of mathematics to society, and a personal approach to the subject seem like esoteric kinds of knowing, I believe that they are key components of MKfT for all mathematics teachers: primary teachers, seconday teachers, tertiary teachers.

There will be a lot to discuss about what "holding mathematics" actually means, and what chance teachers have to develop this kind of knowledge. I look forward to engaging in such discussions with many people from many different areas of mathematics education.

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