It Seems to Matters Not Whether it is Partitive or Quotitive Division When Solving One Step Division Problems

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This paper reports on strategies 26 Grade 3 students used to solve a range of division word problems in a one-to-one interview. The focus is on the strategies used by the students to solve partitive and quotitive division problems pertaining to four different semantic structures. Of particular interest was the range of strategies used for each form of division. Results suggest that there was little difference between the strategies used for partitive and quotitive division.

Studies on children's solutions to division problems indicate that children as young as five can solve a variety of problems by combining direct modelling with counting and grouping skills, and with strategies based on addition and subtraction (e.g., Correa, Nunes & Bryant, 1998; Kouba, 1989). While this may be true, to understand division requires more than knowledge of sharing out a collection equally: it requires an awareness of the relationship between the dividend, divisor and the quotient, and the role of each in a division problem (Correa et al., 1998).

To appreciate the complexity associated with the learning of division it is necessary to consider how distinctly different it is from other operations. Division may be interpreted in two different ways, namely division by the multiplier (partition division) and division by the multiplicand (quotitive division). In partition division (commonly referred to as the sharing aspect), the number of subsets is known and the size of the subset is unknown, whereas in quotition division (otherwise known as measurement division), the size of the subset is known and the number of subsets is unknown (Fischbein, Deri, Nello & Merino 1985; Greer, 1992). The division problem 12 ÷ 4 could be interpreted as a partitive problem, such as: 'Twelve lollies are shared equally among 4 children. How many did they each receive?' Interpreted as a quotitive problem, using the same context, it would be: 'There are 12 lollies and each child receives 4. How many children will receive lollies?' While the quotient is the same for each, the model is quite different. These examples illustrate the fact that the role of the size of the portion and the number of recipients actually reverses in partitive and quotitive division (Squire & Bryant, 2002).

Studies on children's solution strategies to division one step word problems indicate that children generally begin with direct modelling and unitary counting, progress to skip counting, double counting, repeated addition or subtraction, then to the use of known multiplication or division facts, commutativity and derived facts (e.g., Mulligan, 1992; Mulligan & Mitchelmore, 1997). Kouba (1989) found children used two intuitive strategies when solving quotition problems: either repeated subtraction or repeatedly building (double counting and counting in multiples). For partitive division, they drew on three intuitive strategies: sharing by dealing out by ones until the dividend was exhausted; sharing by repeatedly taking away; and sharing by repeatedly building up. However, Brown (1992) reported that children in Grade 2 tended to solve partitive problems using grouping strategies, rather than sharing strategies, but the strategy did not always correctly model the action of the problem. Murray, Oliver and Human (1992) found the children's solution strategies (Grades 1 to 3) for partitive and quotitive division problems initially modelled the problem structure. As the children became experienced they were more flexible in the strategies they used and ignored whether it was a partitive or quotitive

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problem. Their study differed to others in that the emphasis was on developing the meaning of division and solution strategies using a problem solving approach.

This study differs from previous studies on division at Grade 3, as the focus was on students' solution strategies for partitive and quotitive division word problems, pertaining to four different semantic structures (equivalent groups, allocation/rate; rectangular arrays, *times as many*). The excerpts of children's solution strategies indicate an intuitive use of multiplication for solving both partitive and quotitive division problems. This was evident in studies of students in Grades 4 to 6, involving numbers beyond the multiplication fact range (Fischbein et al., 1985; Heirdsfield et al., 1999). Unlike earlier studies where the children used physical materials (counters) and drawings to solve the problems, the students in this study were encouraged to solve them mentally.

Methodology

This paper draws on one of the findings of a larger study conducted from March to November 2007, of young children's development of multiplicative thinking. The study involved students aged eight and nine years from two Grade 3 classes in two primary schools in a middle class suburb of Melbourne. A sample of 13 (case studies) representing a range of performance from each grade was selected using a one-to-one task based interview. Following a teaching experiment on division, over a three-week block, in October, the researcher administered a one-to-one, task-based interview to each case study student in November. The purpose of the interviewing was to gain insights into and probe students' understanding of multiplicative structures and strategies used in division problems.

Instruments

The main sources of data collection were interviews. The researcher developed a one-to-one, task-based interview on division, consisting of two problems (partitive and quotitive) for each semantic structure identified by Anghileri (1989) and Greer (1992): equivalent groups, allocation/rate, arrays, and times as many. For each problem there were three levels of difficulty, rated as easy, medium or challenge, from pilot testing.

The division interview consisted of eight division word problems (see Table 1) devised using the multiplication word problems from the earlier interview. Each category included both a partitive (sharing) and quotitive (measurement) problem to identify whether there was a relationship between the strategies students chose and the division type.

Interview Approach

The case study students in both schools were interviewed 3 weeks after the 15-day classroom intervention. Each interview was audio taped and took approximately 30 to 45 minutes, depending on the complexity of the student's explanations. Responses were recorded and any written responses retained. The problems were presented orally, and paper and pencils were available for students to use at any time. Generous wait time was allowed and the researcher asked the students to explain their thinking and if they thought they could work the problem out a quicker way. Students had the option of choosing the level of difficulty to allow them to have some control and feel at ease during the interview. If a student chose a challenge problem and found it too difficult, there was an option to choose an easier problem.

Table 1
Division Interview Whole Number Word Problems

Semantic structure	Aspect of division	Problem
Equivalent groups	Partition	I have 48 cherries to share equally onto 3 plates. How many cherries will I put on each plate? (M 18, 3; E 12, 3)
	Quotition	72 children compete in a sports carnival. Four children are in each event. How many events are there? (M 24, 4; E 12, 4)
Allocation/ Rate	Partition	I rode 63 kilometres in 7 hours. If I rode at the same speed the whole way, how far did I ride in one hour? (M 28, 7; E 15, 5)
	Quotition	I have 90 cents to spend on stickers. If one packet of stickers cost 15 cents how many packets of stickers can I buy? (M 60, 5; E 30, 5)
Rectangular Arrays	Partition	One hundred and two pears are packed into the fruit box in 6 equal rows. How many pears are in each row? (M 54, 6; E 24, 4)
	Quotition	I cooked 84 muffins in a giant muffin tray. I put 6 muffins in each row, of the tray. How many rows of muffins on the tray? (M 36, 4; 4; E 20, 4)
Times as many	Partition	Sam read 72 books during the readathon, which was 4 times as many as Jack. How many books did Jack read? (M 36, 4; E20, 4)
	Quotition	The Phoenix scored 48 goals in a netball match. The Kestrels scored 16 goals. How many times as many goals did the Phoenix score? (M 28,7; E 18, 6)

Method of Analysis

Initially, the researcher coded the students' responses as correct, incorrect, or non-attempt as well as the level of abstractness of solution strategies, drawing upon the categories of earlier studies (Anghileri, 2001; Kouba, 1989; Mulligan, 1992; Mulligan & Mitchelmore, 1997). Those listed and defined in Table 1 according to the level of abstraction; include direct and partial modelling, repeated addition or subtraction, building up, doubling and halving, multiplicative calculation and wholistic thinking. For the purpose of this paper, the term abstraction refers to a student's ability to solve a problem mentally without the use of any physical objects (including fingers), drawings or tally marks. If a student used a strategy that reflected lack of understanding of the task, this was coded as an unclear strategy.

The strategies presented in Table 2 are in a hypothetical order of sophistication, moving from concrete modelling to levels of abstraction (a student's ability to solve a problem mentally without the use of any physical objects, drawings or tally marks).

Table 2
Solution Strategies for Whole Number Division Problems

Strategy	**Definition**
Unclear	Strategy reflects lack of understanding of task, or is unrelated to task.
Direct Modelling	Uses sharing or one to many grouping with materials, fingers or drawings and calculates total by skip or additive counting.
Partial Modelling	Partially models situation with concrete materials, or drawings using sharing or one to many grouping. Consistently uses skip or double counting to find the total.

Repeated Addition or Repeated Subtraction	Repeatedly adding multiples of the divisor from zero until reaches the dividend, or subtracting multiples of the divisor from the dividend until reaches zero. Partial drawing/recording in some instances, if unable to fully coordinate the two composite units.
Building up	Skip counts using the divisor up to the dividend, without the use of any drawing or tally marks.
Doubling and Halving	Derives solution using doubling or halving and estimation, attending to the divisor and dividend. Recognises multiplication and division as inverse operations.
Multiplicative Calculation	Automatically recalls known multiplication or division facts, or derives easily known multiplication and division facts, recognises multiplication and division as inverse operations.
Wholistic Thinking	Treats the numbers as wholes—partitions numbers using distributive property, chunking, and or use of estimation.

Results and Discussion

The frequencies of strategies used by the combined cohort on the different whole number partition and quotition division problems are presented in Tables 3 and 4. For each of the semantic structures the level of difficulty was included to provide an indication of whether there was any relationship between the choice of strategy and the numbers in a given problem, or whether the semantic structure of the problem influenced the choice of strategy. The thick black line delineates the strategies in the following way: those left of the line involve some form of modelling, whereas those to the right of the line refer to the use of multiplicative thinking as the students were using some degree of abstraction.

The following codes are used in the tables: Unclear strategy (US), Direct modelling (DM), Partial modelling (PM), Repeated addition (RA), Repeated subtraction, (RS), Building up (BU), Doubling or halving (DH), Multiplicative calculation (MC), Wholistic thinking (WT).

Table 3
Strategy Use Across Partition Division Problems (26 students)

Semantic structure	Level of difficulty	US	DM	PM	RA or RS	BU	DH	MC	WT
Equivalent groups	Easy								
	Medium		3	6		3		1	
	Challenge			1			1	8	3
Allocation/ rate	Easy	1	2			1			
	Medium			1	1	4			
	Challenge					3		13	
Rectangular array	Easy		2	1		1		1	
	Medium		1	1		2		5	
	Challenge		1			1	1	7	2
Times as many	Easy	6			1	1		1	
	Medium				3	2		2	
	Challenge				1		3	3	3
Total		7	9	10	6	18	5	41	8

From Table 3, it can be seen that the most commonly chosen of these strategies was multiplicative calculation overall, especially for the allocation/rate challenge problem. In fact, multiplicative calculation and wholistic thinking were the preferred strategies by those students who chose the challenge problems. One could infer from this that when presented with problems outside their multiplication fact range students draw on more efficient strategies using their problem solving skills and number sense.

An unexpected result was the number of students who used a modelling strategy for the following medium equivalent groups problem, ‘I have 18 cherries to share equally onto 3 plates. How many cherries will I put on each plate?’ Of the thirteen who chose this problem, nine used some form of modelling strategy, yet these same students used multiplicative strategies for other problems. One might infer from this that either the word ‘share’ or the numbers used prompted the strategy choice.

A surprising finding relates to the strategy use for the *times as many* problems compared to the equivalent group problems. A higher proportion of students (77%) used multiplicative strategies for the *times as many* problems compared to those who used multiplicative strategies for the equivalent groups problems (61%). This was unexpected given the students were more familiar with the equivalent group problems and the *times as many* problems are considered to be more difficult (Anghileri, 1989; Greer, 1992), as the semantic structure is quite different to that of equivalent groups, allocation/rate and rectangular arrays. Lastly, it is worth noting the use of the repeated subtraction strategy, which, as stated earlier in the literature review, was more commonly used for quotitive problems. Having said that, five students used this strategy for the *times as many* problems, the language of which may have prompted its use.

Table 4
Strategy Use Across Quotition Division Problems (26 students)

Semantic structure	Level of difficulty	US	DM	PM	RA or RS	BU	DH	MC	WT
Equivalent groups	Easy								
	Medium		4			6			
	Challenge			1		1	4	10	
Allocation/ rate	Easy		1						
	Medium		1	1		5	1	1	
	Challenge			1		6	3	6	
Rectangular array	Easy		2	1				1	
	Medium		1	4		5		1	
	Challenge						1	7	3
Times as many	Easy	5				1			
	Medium	4				4			
	Challenge	1				2	4	3	2
Total		10	9	8	0	30	13	29	5

From Table 4, it can be seen that multiplicative calculation and building up were the most popular choice of strategies overall. In fact, of those who chose the medium problems, just over half (53%) used the building up strategy. One might infer from this that students tend to revert back to less efficient strategies when solving an easier problem.

It was surprising that multiplicative calculation was the preferred strategy in only the equivalent groups and rectangular array challenge problems, given that the six students who used the building up strategy for allocation/rate problems used multiplicative calculation for at least four of the eight problems overall. One might infer from this that the students’ choice of solution strategies is influenced by the semantic structure of the problem. This was also true for the *times as many* challenge problems, as six students who chose to use either the building up or doubling/halving strategies consistently used either multiplicative calculation or wholistic thinking to solve other problems. Further, it is worth noting that no students used repeated subtraction. This was surprising given it is one of the strategies generally associated with quotition division (Kouba, 1989).

Table 5
Frequency of Strategy use for Partitive and Quotitive Problems (26 students)

Task type	US	DM	PM	RA or RS	BU	DH	MC	WT
Partitive	7	9	10	6	18	5	41	8
Quotitive	10	9	8	0	30	13	29	5

From Table 5, it can be seen that just over three quarters of the cohort used multiplicative thinking for both forms of division, which is significant, given the difficulty of some of the challenge problems. Second, a higher proportion of students used the building up strategy for quotitive problems than for partitive problems. A possible explanation for this is the nature of the numbers in the quotitive problems and that perhaps the students found the allocation/rate and rectangular array partitive problems less demanding than the corresponding quotitive problems. Third, a similar number of students used some form of modelling for both forms of division. Fourth, a similar number of students used an unclear strategy for the *times as many* problems for both forms of division. In fact, in both instances these students found the difference between the dividend and the divisor indicating they had no understanding of this type of problem.

Lastly, it is worth noting that of the 26 students interviewed, 20 were able to explain what kind of problem it was and how they knew. Some indicated that division was not that hard if you knew multiplication because it is just the opposite of division.

A closer look at the students' solution strategies indicate that in fact many ignored whether it was partitive or quotitive division and solved the problems using the inverse operation. In each instance the students were able to record a division number sentence and link the solution back to the context of the problem.

The following abridged excerpts of four students' solution strategies for both partitive and quotitive rectangular array word problems, indicate their ability to think flexible and use their understanding of multiplication to solve a range of division problems.

RA-P: One hundred and two pears are packed into the fruit box in 6 equal rows. How many pears are in each row?

Jules: Twelve sixes are 72 and 15 sixes are 90 so 17 sixes are 102.

Sharn: 20 sixes is 120, but that's too much. 15 sixes is 90 and another 12 is 102, so that's 17 sixes and 17 pears in each row.

Bindy: Double six is twelve, so that's 2, double 12 is 24, so that's 4, double 24 is 48, so that's 8, double 48 is 96 so that's 16 and another 6 is 102. So there are 17 pears in each row.

Mark: Twelve sixes are 72, 26 sixes is 144, 20 sixes is 120 but that's 18 too much, so it's 17 sixes because you take away 3 sixes, from 20 sixes

RA-Q: I cooked 84 muffins in a giant muffin tray. I put 6 muffins in each row, of the tray. How many rows of muffins on the tray?

Jules: 12x6 is 72, and another 12 would make 84, that's another 2 lots of 6 so 14 sixes is 84

Sharn: 10 rows of 6 is 60, 24 left that's 4 x 6, so 14x6 = 84, so that's 14 rows of muffins.

Bindy: I know 6x10 is 60 and 6x4 is 24, so 6x14 is 84, or 14 rows of muffins.

Mark: 12 sixes are 72, 13 are 78, so 14 are 84. So there are 14 rows of muffins.

In both instances Jules and Mark used multiplicative calculation and Sharn used wholistic thinking, whereas Bindy used doubling and adding one more multiple of 6 for the partitive problem and wholistic thinking for the quotitive problem. It appears that the students did not distinguish between partitive and quotitive as they did not think '6 times something' for the partitive problem. Instead they attended to the numbers and their knowledge of multiplication facts.

A similar pattern emerged when analysing the students' strategies for solving the *times as many* partitive and quotitive word problems as illustrated in the following excerpts.

> TM-P: Sam read 72 books during the readathon, which was 4 times as many books as Jack. How many books did Jack read?
>
> Bindy: 6x12 is 72 and 6x6 is 36 and half of 72 is 36 and because it's 4 times I need to halve 36 and that's 18, so that means 4 x18 is 72 so Sam would read 18 books.
>
> Mark: 12x6 is 72, but that's 6 times as many and it has to be something times 4 is 72. 12x4 is 48 and double that would be 24x4, which is 96. I still think the doubling strategy could work. 8x4 is 32, 16x4 would be 64, 20x4 is 80 but that's too much so it has to be between 12x4 and 20x4. 20x4 is 2x4 too much so I need to take that away so it's 18x4, so Sam read 18 books.
>
> Sandy: 4 times something equals 72. 72 take away 40 (4 times 10) is 32, and 32 is 4x8. 4x20 is 80, but that's too much. I know 6x12 equals 72, but 12 would be 6 times not 4. I need to take 8 off 80 to get 72 so it would be 18, because 4x10 is 40 and 4x8 is 32 and together that's 72.
>
> TM-Q: The Phoenix scored 48 goals in a netball match. The Kestrels scored 16 goals. How many times as many goals did the Phoenix score?
>
> Bindy: I doubled 16 to 32 and knew I needed another 16 to 48 so it's 3 times as many goals.
>
> Mark: I counted by 16s. 16, 32, 48. It's the same as counting by 8s but you take every second one. I could have said 6 eights are 48 and halved the 6 and doubled the 8 which is 3x16.
>
> Sandy: 10 and 30 is 40. 6 and 2 is 8. Thirty–two is two times 16 so it would be 3 times because another 16 makes 48.

In the partitive problem Bindy and Mark used known facts as a staring point, Mark used commutativity, doubling and trial and error, whereas Bindy used halving and kept the focus of '4 times' in her mind. Sandy used more wholistic thinking by splitting the product and using the distributive property. In the quotitive problem Sandy again split the product up into known parts as a starting point, whereas Bindy used doubles and realised 48 would be 3 multiples of 16. Mark suggested an alternative doubling and halving strategy, when asked if he could do it a faster way.

Although different strategies have been used to solve these problems, they reflect some form of multiplicative thinking. These examples illustrate the students' confidence and ability to mentally manipulate numbers using number sense. This support the findings of Murray et al., (1992) of providing students in these early grades with both partitive and quotitive division problems using a problem solving approach, to allow students to develop a range of efficient mental strategies.

Conclusion

The findings presented in this paper suggest that the semantic structure or the range of numbers in the problem influences students' solution strategies rather than whether the problem is partitive or quotitive. Second, having an understanding of multiplication supports students' development of division and enables students to use the inverse operation to solve division problems, as indicated by the students' preference to use multiplication to solve both partitive and quotitive division problems. Third, students need a variety of experiences a range of semantic structures and contexts to understand fully the operations of multiplication and division. This finding resonates with the work of Mulligan and Mitchelmore (1997) but this study adds to the body of knowledge pertaining to students' solution strategies to partitive and quotitive division for a range of semantic structures, and presents a strong case for linking multiplication to the teaching of division.

References

Anghileri, J. (2001). *Principles and practices in arithmetic teaching.* Buckingham: Open University Press.

Anghileri, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics*, *20*(4), 367-385.

Brown, S. (1992). Second-grade children's understanding of the division process. *School Science and Mathematics, 92*(2), 92-95.

Burton, G. (1992). Young children's choices of manipulatives and strategies for solving whole number division problems. *Focus on Learning Problems in Mathematics, 14*(2), 2-17.

Correa, J., Nunes, T., & Bryant, P. (1998). Young children's understanding of division: The relationship between division terms in a noncomputational task. *Journal of Educational Psychology, 90*(2), 321-329.

Fischbein, E., Deri, M., Nello, M.S., & Marino, M.S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3-17.

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 276-295). New York: Macmillan.

Heirdsfield, A. M., Cooper, T. J., Mulligan, J., & Irons, C. J. (1999). Children's mental multiplication and division strategies. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 89-96). Haifa, Israel: PME.

Kouba, V. L. (1989). Children's solution strategies for equivalent set multiplication and division problems. *Journal for Research in Mathematics Education, 20*(2), 147-158.

Mulligan, J. (1992). Children's solutions to partition problems. In B. Southwell, R. Perry, & K. Owens (Eds.), *Proceedings of the 15th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 410- 420). Sydney: MERGA.

Mulligan, J. T., & Mitchelmore, M.C. (1997). Identification of multiplicative thinking in children in Grades 1-5. *Journal for Research in Mathematics Education, 28*(3), 309-331.

Murray, H., Oliver, A., & Human, P. (1992). The development of young students' division strategies. In W. Geeslin & K. Graham (Eds.), *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 152-159). Durham, New Hampshire: PME.

Squire, S., & Bryant, P. (2002). From sharing to dividing: Young children's understanding of division. *Developmental Science, 5*(4), pp. 452-466.