Revealing Conceptions of Rate of Change

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Rate of change is an important mathematical concept. Research referring to students' difficulties with this concept spans more than twenty years. Research suggests that problems experienced by some calculus students are likely a result of pre-existing limited or incorrect conceptions of rate of change. This study investigated 23 Victorian Year 10 students' understanding of rate as revealed by phenomenographic analysis of interviews. Eight conceptions of rate of change emerged. Four important aspects of the concept were identified and gaps in students' thinking defined. In addition, the employment of phenomenography, to reveal conceptions of rate, is described in detail.

Introduction

Rate of change is an important mathematical concept, but even many calculus students find it particularly troublesome (Orton, 1983; Ubuz, 2007). Robert and Speer (2001) suggest that it may be that the problems experienced by some calculus students are a result of pre-existing conceptions of foundation concepts such as rate of change. But what are the conceptions of rate of change held by Year 10 students and how do they differ from important aspects of rate of change which are required for students to develop a the mathematical concept as it is generally understood and accepted by experts?

This paper reports on a study in which phenomenographic analysis of student interviews was used to identify conceptions of rate of change. The full spectrum of students' conceptions revealed in these interviews is presented in the form of an *outcome space* (shown in Figure 2 and discussed below). This is followed by an analysis of these conceptions to identify aspects which appear critical in assisting students' to correct and extend their understanding of rate of change and so better match the generally accepted concept. Particular emphasis, in this paper, is placed on the description of the use of phenomenography in revealing the conceptions of rate held by Year 10 students.

Background

Quantitative methods, such as surveys and testing, have previously been used to examine calculus students' understanding of rate of change (see for example Orton, 1983; Ubuz, 2001). These quantitative methods have proved to be valuable in highlighting the difficulties, with the concept of rate of change, experienced by some calculus students. Quantitative methods give numeric measures based on statistical analysis of responses to items testing a predetermined hypothesis. Quantitative research "gives a broad, generalizable set of findings" resulting from "the use of standardized measures" fitting the "varying perspectives and experiences of people … into a limited number of predetermined response categories to which numbers are assigned" (Patton, 2002, p.14). At the outset of this study, no such predetermined categories, of Year 10 students' conceptions of rate of change, could not be found for use as the basis for a quantitative study. Some research relating to students' understanding of rate of change has involved case studies (see for example Zandieh, 2000), whereas other investigations have employed mixed methodologies, combining quantitative measures with some form of qualitative analysis. In that case qualitative data is examined for the degree of support it gives for a pre-determined hypothesis (see for example Rowland & Jovanoski, 2004). A fresh approach seemed

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appropriate in order to try to identify and categorise the conceptions of rate of change held by Year 10 students and so better understand the prior knowledge such students may bring to a study of calculus. The nature of the research question guides the choice of methodology: consequently, for this study, an in depth investigation of students' conceptions was required. This suggested that a qualitative methodology would be appropriate and there are many qualitative methods from which to choose. Phenomenography was trialled as a methodology in this study because of its demonstrated efficacy in research in a range of other educational contexts (see for example Marton, Runesson, & Tsui , 2004) and in particular it has previously been used successfully in mathematics education, for example, by researchers investigating students' conceptions of equivalence relations, (Asghari & Tall, 2005). The following sections outline the theoretical framework and then describe the application of phenomenography in this study.

Theoretical Framework

"Phenomenography is an interpretive research approach that seeks to describe phenomena in the world as others see them, the object of the research being variation in ways of experiencing the phenomenon of interest" (Bruce, Buckingham, Hynd, McMahon, Roggenkamp, & Stoodley, 2004, p.147), for example, the concept of rate of change as in this study. It is important to emphasise that a phenomenographic researcher attempts to describe a phenomenon as others see it, not as it is seen by the researcher. As will be explained below, the aim of phenomenography is to reveal *categories of description* delineated by the *dimensions of variation* which emerge from the data and hence, structure the categories into an *outcome space*.

The process of phenomenographic analysis begins with tentative categories which emerge from the data in much the same way as themes are identified from the data in a grounded theory approach. The categories or themes emerge from the analysis of the transcripts rather than the researcher forcing the data to fit into a pre-determined model. Typically, transcriptions of interviews are examined to find ways of grouping the participants' responses into these *categories of description* according to the features and characteristics which they hold in common. These features and characteristics are referred to as the *dimensions of variation*. Each category is carefully described in terms of the dimensions of variation so that any particular response statement may be classified into just one of the categories.

Meaningful quotes, relating to the phenomenon, are taken from the individual transcripts and pooled, shifting attention from the individual to the meanings expressed by the group as a whole (Marton, 1988). Phenomenographic analysis requires a series of iterations to refine the categories through repeated reading of the transcripts (Akerlind, Bowden & Green, 2005). An examination of the early tentative categories and further interrogation of the data establish tentative dimensions of variation. These dimensions delineate the categories one from another according to the number of dimensions or values of dimensions which are evident in each category (Cope, 2000). However, the categories and dimensions are so interrelated that it is difficult to discuss one without reference to the other. The initial categories suggest the initial dimensions and the data is repeatedly re-examined to refine both the categories and the dimensions. The dimensions facilitate an ordering of the categories into an *outcome space* where each category is placed in relationship to the other categories. The hierarchical nature of the outcome space infers that some categories of description indicate a higher level of perception of the phenomenon in question (Marton, 1988). The outcome space is considered to be the final result of a phenomenographic investigation.

For a phenomenographic study "the validity of the outcomes is related to the processes that are used at all stages of phenomenographic research" (Akerlind et al., 2005, p. 89), that is: the selection of the sample for maximum variation; the conduct of consistent, open interviews; and the iterative analysis which remains focused on the meanings expressed in the transcripts taken as a whole. Replication of the outcome space is not considered necessary, but other researchers should be able to recognise the categories of description (Marton, 1988).

Method

The decision to employ phenomenography to investigate conceptions of rate of change directly influenced the manner of data collection. Rigorous guidelines have been established about the conduct of phenomenographic interviews, where the researcher endeavours to gain as much information as possible about each participant's conceptions, for later analysis.

The participants of a phenomenographic study are chosen in order to maximise the variation in perceptions of the phenomenon, rather than to provide a statistically representative sample (Akerlind et al., 2005). The participants in this study were twenty-three Year 10 students. These students were appropriate participants for this research because, according to the Victorian Essential Learning Standards (VCAA, 2007) these participants, in Year 10, may be expected to have previously studied constant rate and some functions where rate varies, such as quadratic, exponential and logarithmic functions. It was expected that they would provide a variety of responses because they attended five different schools that differed in the following ways: state schools; independent schools; girls' schools; co-educational schools; regional schools; schools in the outer-suburban fringe of Melbourne, inner-city and suburbs with a high level of cultural diversity.

In phenomenographic research, probing interviews are usually conducted and transcribed from the audio-record in as much relevant detail as possible. In this study, video was used for data collection because the combination of utterances and gestures have the potential to give greater insight into students' thinking (see Herbert & Pierce, 2007 for more detail about the collection and interpretation of video data for phenomenographic analysis of the interview data of this study).

The video-recorded interviews consisted of a discussion of interactive computer simulations of real-world contexts involving rate of change: a window partially covered by a blind, in Geometer's Sketchpad; and characters walking, in JavaMathWorld. Each simulation showed a real world context and multiple mathematical representations of the functions involved in the simulations. The video recordings of the interviews were digitised and examined, along with the transcriptions, for evidence of conceptions of rate of change. Phenomenographic analysis, consisting of the refinement of the categories through an iterative process involving repeated reference to the interview transcripts and videos, was then undertaken.

Results

Category	Brief Description	Illustrative Quotes
A	Rate is experienced as a word rating a quality.	**R:** Can you give me an example? **I20:** ratings on movies and stuff
B	Rate is experienced as a word associated with a numeric value.	**R:** what is rate? [pause] can you give me an example of rate? **I3**:Well there is that bank rate thing **R:** Can you tell me more about that? **I3:** Well they add rate if you don't pay your bills or something on time on time, they keep adding.
C	Rate is experienced as the result of a formula calculation with little meaning.	**R**: What is rate? [pause] Can you give an example of rate? **I2**: if you gave me a problem [formula], I could substitute the numbers in it and then I would be able to give you a rate
D	Rate is experienced as a single quantity.	**R**: tell me about the rate the area is changing. **I12** : It's getting greater. **R**: What's getting greater? **I12**: Like, the area of sunlight's getting greater **R**: can you tell me about the rate that the clown is walking? You can make him walk as often as you like. **I14**:He's walking at about 8 seconds.
E	Rate is experienced as a relationship between two changing quantities.	**I3**: As the area of sunlight increases, the height of the blind also increases. **R**: So can you tell from the table who's walking faster? **I6**: You can tell the clown has walked, um, more quickly because his distance is greater for the period of time.
F	Rate is experienced as a constant numeric relationship between two changing quantities.	**R**: what makes you think he is walking faster? **I3**: Because the metres that he is walking is bigger rather than being smaller [pause] with the frog the less metres that he is covering [pause] the clown is going in 1 meter he's reached like 5, I mean 1 second he's sort of reached 5 metres rather than the frog in one metre he is going 3 metres, I mean one second he's reached 3 meters. **I7**: the area increases by a rate of 3.2 for every 0.5.
G	Rate is experienced as a numeric relationship between two changing quantities of distance and time i.e. speed.	**R**: What is rate? **I5**: [pause] Like how fast things are going? **I10**: He's walking about four meters per second…the rate he is walking remains the same. **I1:** he [the clown] walks a 22 meters in 7 seconds and then he [the frog] only makes 7 meters in 7 seconds.
H	Rate is experienced as a numeric relationship between any two changing quantities	**I4**: That's increasing [pause] so that it [area] starts off small 1 and it goes up to 2.6, so that's 1.6, the difference, and then the next one is 2.2 the difference and the one after that is 2.8 the difference

Figure 1: Brief descriptions of categories with illustrative quotes

The analysis, described above, resulted in the constitution of the outcome space seen in Figure 2. It consists of eight categories that are illustrated in Figure 1, above, by examples of excerpts from student interviews. During analysis of the data, four dimensions of variation emerged: *Focus on word 'rate'*; *Focus on variables*; *Focus on relationship between variables*; and *Focus on nature of related variables*. The dimensions formed the basis of discriminating between the categories by identifying which dimensions and which values are in focus for each category. For example, in Category B, only awareness of rate as a numeric quantity of the dimension of variation *Focus on word 'rate'* is discerned. No other dimensions or their values are recognised.

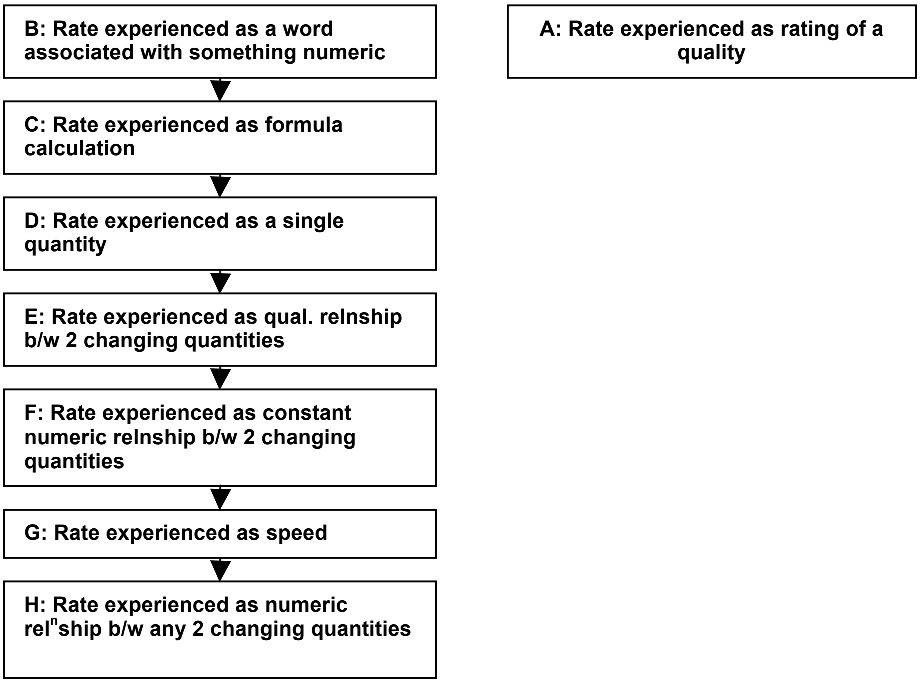

Figure 2: The Rate Outcome Space

The relationship between the categories of description and the dimensions of variation is illustrated on Figure 3. The shading indicates, for example, that the first dimension *Focus on word 'rate'* may take values 'quality' and 'quantity'. These values differentiate between the conceptions where 'rate' is experienced as a numeric quantity or rating of a quality. This dimension reflects an emphasis on the word 'rate' and whether the word brings awareness of rate as a numeric quantity. The second dimension *Focus on variables* may take values 'one-fixed', 'one-changing', and 'two-changing' and refers to the number and type of variable. These values differentiate between the conceptions where 'rate' is experienced as a static quantity or a changing quantity. The discernment of two changing variables is a critical aspect in the discernment of rate as a relationship. This dimension illustrates the bridge from 'incorrect' conceptions of rate to 'correct' conceptions of rate.

The third dimension *Focus on relationship between variables* may take values 'qualitative', 'quantitative-constant', 'quantitative-variable' and refers to the nature of the description of the relationship between the variables. These values differentiate between the conceptions where 'rate' is experienced as a relationship between two changing quantities. This dimension separates the different 'correct' but incomplete conceptions of rate. The fourth dimension *Focus on nature of related variables* may take values 'distance and time' and 'any pair' and refers to the context in which the concept of rate is explored. These values differentiate between the conceptions where the context of the rate differs. This dimension separates 'correct' situation-dependant conceptions of rate from 'correct' abstract conceptions of rate which are independent of any particular real-world setting. Figure 3 summarises the dimensions and their values which are focal in awareness for each category.

		Categories of Description							
Dimension	**Value**	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**
Word 'rate'	Quality	■							
	Quantity		■	■	■	■	■	■	■
Variables	One - fixed			■					
	One - changing				■				
	Two - changing					■			
Relationship	Qualitative						■	■	■
	Quantitative constant						■	■	■
	Quantitative variable							■	■
Nature of Variables	Distance & time							■	■
	Any pair								■

Figure 3: Summary of values of dimensions discerned in each category

Discussion and Implications

The outcome space may provide a guide to determining the current condition of an individual student's conception of rate, which may suggest strategies to move their conception towards a more correct or complete concept of rate of change. Determining which aspects of a concept are missing or defective involves the comparison of the generally accepted conception of the phenomenon with the conceptions of the phenomenon held by the learners (Cope, 2000). So, this means a comparison of the conceptions of rate of change found in the outcome space with the ideal conception of rate generally accepted by experts. Such a conception encompasses the notion of instantaneous rate as the gradient of the tangent to a curve at a given point and is developed from the notion of average rate or the gradient of the straight line joining two points on a curve. It subsumes the notion of constant rate as a special example where the instantaneous rate is the same for all points on a curve, in this case a straight line. At its most abstract, rate of change can be applied in any real-world or abstract context. Rate of change is often synonymous with derivative.

Each category of description in the outcome space is compared with the 'expert's' conception of rate to determine which aspects of the students' understanding need to be explicitly addressed in classrooms to progress the development of the concept. Figure 4 provides a list of the categories, in reverse order, and identifies any aspects missing from the category when compared with an expert's conception. As the categories are structured hierarchically in the outcome space, aspects missing from one category are also missing from the next. For example, 'Independence from setting' and 'Abstraction' are missing from Category G, so these aspects are also missing from Category F in addition to 'Variable rate'. Categories E, F and G may be considered as mathematically correct, but incomplete, whereas Categories A to D show various degrees of incorrectness. This comparison facilitates the identification of gaps that need targeted teaching and may require the development of suitable teaching material and learning activities to specifically address these aspects. Rate must be experienced as numeric and not a rating and the discernment of two changing quantities is necessary before a relationship between them can be appreciated. Such a relationship is crucial in experiencing rate as experts experience it, so careful attention to establishing rate as a relationship between two changing quantities is essential in development of the concept of rate.

Once this relationship assumes prime importance in a student's thinking about rate, then experiences with different kinds of rate may facilitate a more complete conception of rate.

Category	Additional Missing Element(s)
H	None
G	Independence from setting Abstraction
F	Variable rate
E	Quantification
D	Two variables Relationship
C	Change
B	Calculation
A	Numeric association Something to do with maths

Figure 4: Missing elements in students' conceptions of rate of change

Another important aspect of the concept of rate of change is the possibility that rate may vary. Differentiation involves working with variable rate, but this is difficult unless a student has some idea what this is, even if that idea can only be articulated generally with descriptive words. The notion of rate expressing a measure of the relationship between two quantities is essential, so quantification of rate is critical. Finally, the applicability of rate, to any context where a relationship exists between two changing quantities, is also significant in an expert's concept of rate of change. These important aspects of rate of change have been summarised in Figure 5

1. rate as a relationship between two changing quantities.
2. rate as a relationship between two changing quantities which may vary.
3. rate as a numerical relationship between two changing quantities which may vary.
4. rate as a numerical relationship between any two changing quantities which may vary.

Figure 5: Important aspects of the concept of rate of change

Conclusions

This phenomenographic study revealed eight conceptions of rate of change held by the Year 10 students interviewed towards the end of school year. The final result of the study is the outcome space of Year 10 students' conceptions of rate of change, with accompanying description of the variation in conceptions. The resulting outcome space now provides the pre-determined categories that might be appropriate for use in a large scale study to determine the prevalence of each category. Phenomenography proved to be an effective methodology to provide evidence for students' conceptions of rate of change. Comparison, of conceptions of rate revealed by this study with the conception commonly accepted by experts, highlighted aspects of rate of change which are vital in the development of a robust concept of this phenomenon but missing or incomplete in the students' conceptions. These important aspects are: awareness of a relationship between two changing quantities; awareness that rate may vary; and facility to quantify both constant and variable rate in any context.

The results of phenomenographic research may inform curriculum designers and teachers and so assist them to prepare material which guides middle-years students on a path from lower to higher levels of conceptions shown in the outcome space. In addition, this information may assist teachers of calculus, curriculum designers and textbook authors to prepare material for calculus students which takes into account the possible state of their students' initial conceptions of rate of change.

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