

Making Sense of Partitive and Quotitive Division: A Snapshot of Teachers' Pedagogical Content Knowledge

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Responses to an item intended to assess teachers' Pedagogical Content Knowledge (PCK) are discussed. The item examined 92 teachers' capacity to name the two forms of division, provide a simple representation and story problem for each, and explain which form helps to make sense of dividing a whole number by a decimal. Teachers appeared more familiar with partitive than quotitive division. Apparent understanding of quotition with whole numbers did not always transfer to division with decimals. Some findings from the broader PCK research are also shared.

Theoretical Background

Since Shulman (1986) first identified the more general pedagogical content knowledge that all teachers bring to their profession, researchers have attempted to conceptualise and measure teachers' *mathematical knowledge for teaching* (Ball & Bass, 2000; Chick, 2007; Hill, Ball, & Schilling, 2008; Hill, Sleep, Lewis, & Ball, 2007). Research has determined that student outcomes can be improved by enhancing teachers' PCK (Hill, Rowan & Ball, 2005). Initiatives that provide professional development that focus on helping teachers shift from a procedural approach to a conceptual approach to teaching mathematics (Cheeseman, 2007; Cooper, Baturu, & Grant, 2006; Watson, Beswick, Brown, & Callingham, 2007) may help facilitate this change in teacher knowledge and practice. Some researchers have investigated teachers' PCK associated with a particular domain of mathematics, such as proportional reasoning (Watson, Callingham, & Donne, 2008), area and perimeter (Yeo, 2008), fractions (Watson, Beswick, & Brown, 2006), chance and data (Watson, 2001) and decimals (Chick, Baker, Pham, & Cheng, 2006), while utilising different instruments of assessment (e.g., multiple choice items, open response items, interviews and classroom observations).

Shulman described PCK as "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, explanations and demonstrations—in a word, the ways of representing and formulating the subject that makes it comprehensible to others" (1986, p. 9). Chick (2007) developed a very detailed PCK framework for mathematics, under three broad categories: clearly PCK, content knowledge in a pedagogical context, and pedagogical knowledge in a content context. Within "Clearly PCK," she included elements such as Appropriate and Detailed Representations and Knowledge of Examples. Shulman's definition and these elements have been helpful when we were constructing items to assess teachers' PCK and are relevant to the focus item in this paper.

Most teachers desire that their students learn mathematics with understanding, although what is taken to be understanding can vary considerably (Skemp, 1976). We find the description of Hiebert and Carpenter (1992) helpful. They suggest that we understand something if we see how it is related or connected to other things we know. They also note that understanding is generative, promotes remembering, reduces the amount that must be remembered, enhances transfer and influences beliefs (pp. 74-77). Few would disagree that without a personal, connected understanding of mathematics, a teacher will find it difficult to support students to make meaningful connections.

Understanding the Partitive and Quotitive (Measurement) Models for Division and Their Relationships to Division by a Decimal

According to the *partitive* model of division, the divisor indicates a whole number of parts or subcollections and the quotient is the size of each part. For example in $12 \div 3 = 4$, twelve is shared into 3 equal groups and there are 4 in each group. In the *quotitive* model (measurement division), the divisor indicates the size of the subset and the quotient is the number of equal sized subsets. For $12 \div 3 = 4$, twelve is divided into groups of 3 and 4 is the number of groups of 3. As Mousley (2000) notes, in partitive division the action is sharing; in quotitive division, the action is partitioning.

Mulligan and Michelmore (1997), in a longitudinal study of Grade 2 and 3 students, found that students possessed several *intuitive models* for division when faced with word problems. They defined these models as “internal mental structures corresponding to a class of calculation strategies” (p. 325). These were direct counting, repeated subtraction, repeated addition, and multiplicative operations. They noted that the poor performance on rational number multiplicative problems among older students (and presumably adults) may have arisen because these students “may not have had the opportunity to develop intuitive models or rational-number multiplication and so may not be aware of the equal-group structure of all multiplicative situations” (p. 328).

Research has demonstrated that children (and adults) hold misconceptions about division with decimals and that these misconceptions generally appear to stem from an incomplete or biased view of division (Fischbein, Deri, Nello, & Marino, 1985; Tirosh & Graeber, 1990). That is, their intuitive model for division is mainly the partitive model. This model is inadequate when the division has a divisor that is less than one. In this case the quotitive model is more helpful. Fischbein et al. (1985), working with Italian students, found that the quotitive model did not become stable “and influential” until around age 14 or 15 (p. 15).

As part of a larger study, we aimed to investigate these questions: What understanding do teachers have with regard to partitive and quotitive division? Do teachers who seem to have a conceptual understanding of quotitive division transfer their understanding to division with decimals less than one? In other words, if teachers understand quotitive division, do they use this understanding to make sense of division of decimals less than one? For example, in the case of $8 \div 0.5$, it makes more sense to consider how many halves in 8, than eight shared between a half.

Method

The Contemporary Teaching and Learning of Mathematics Project (CTLM)¹ is a professional learning and research project involving 11 primary schools in Melbourne (Australia) which are participating in a two-year program with Australian Catholic University, consisting of 12 full days of professional development for teachers (including workshops, professional reading, and between-session activities) and in-classroom support from the research team. The project aims to enhance teacher pedagogical content knowledge and student learning. Student learning over time is being assessed by task-based interviews by trained preservice students (Clarke, Roche, Mitchell, & Sukenik, 2006) with a sample of students at each year level, at the beginning and end of the school year.

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The Development of the Assessment Tool for Teacher Mathematical PCK

In developing the framework which would underpin the development of PCK items, there were four considerations:

- frameworks for mathematics PCK developed by other researchers;
- the mathematical content focus of the CTLM Project in 2008 (whole number, rational number, structure which involves early algebraic thinking including the meaning of the equals sign and properties of number, and working mathematically);
- some of the key aspects of the teacher role on which the project was to focus; and;
- the need to develop a questionnaire which could be completed comfortably in around 40 minutes.

In light of the four considerations above, our current framework has the following components:

Pathways: Understanding possible pathways or learning trajectories within or across mathematical domains, including identifying key ideas in a particular mathematical domain.

Selecting: Planning or selecting appropriate teaching/learning materials, examples or methods for representing particular mathematical ideas including evaluating the instructional advantages and disadvantages of representations or definitions used to teach a particular topic, concept or skill.

Interpreting: Interpreting, evaluating and anticipating students' mathematical solutions, arguments or representations (verbal or written, novel or typical), including misconceptions.

Demand: Understanding the relative cognitive demand of tasks/activities.

Adapting: Adapting a task for different student needs or to enable its use with a wider range of students.

The authors developed three questionnaires, for teachers of Prep to Year 2 (5 to 8 year-olds), Year 3 to 4 and Year 5 to 6, respectively. Each questionnaire involved four to six items, and was intended to reflect the broad content focus of the CTLM professional learning program for those grade levels. They also provide data on teachers' capacities in each element of the framework, with several items addressing more than one component.

The Questionnaire: Pilot Use and Use Within CTLM

Items were piloted with a range of classroom teachers and district mathematical consultants, with adjustments made for clarity, ambiguity, and in response to low inter-rater reliability for some items. The intention was that the same items would be used on the first and last day of the professional development program, in April and November, respectively, in order to show growth in PCK, if any, over time. No opportunity was provided for teachers to revisit the items in the intervening period. Some April items were removed for November because they were too difficult to code (e.g., when almost all responses could be considered reasonable for a given item). Others were removed because they were confusing to teachers. Some new items were added in November, with a view to their full use in 2009, if they proved satisfactory.

The item in Table 1 named "Division Stories" was added in November 2008 to the Prep to 2 and Year 3 to 4 questionnaire as a trial item. The intention is to administer this item (with some minor alterations) to a new cohort of teachers in 2009 who are entering the first year of their participation of the CTLM project. All project teachers in Year Prep to 6 will be given this item in March and November in 2009, to determine if the project has

impacted on their understanding in this area. The results of the trialling of this task in 2008 have enabled the authors to create a rubric for scoring teacher responses. The results of the trial are outlined in this paper.

The Questionnaire Item

The item in Figure 1 was administered to 92 teachers (51 teachers in Year Prep to 2 and 41 teachers in Year 3 to 4) at the end of 2008. In relation to our framework, this task falls into the category of *Pathways* and *Selecting*, as defined above, as it involves selecting appropriate examples or methods for representing a particular mathematical idea and identifying key ideas in a particular mathematical domain. The spacing in the item has been reduced because of space constraints of this paper.

When providing opportunities for students to explore division problems, it is important that they are introduced to both forms of division. In the table below, please

- (a) give a name to each form of division;
- (b) draw a simple picture to represent each form of division; and
- (c) provide an example of a story problem that represents each form of division for the problem $15 \div 3$.

Name the form of division	Draw a simple picture to represent this form of division	Write a story problem that represents this form of division for the problem $15 \div 3$.
i)		
ii)		

d) From (i) and (ii) above, which form of division is most helpful when making sense of $8 \div 0.5$?

e) Please explain why.

Figure 1. Division stories

Results

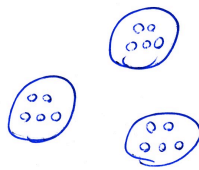
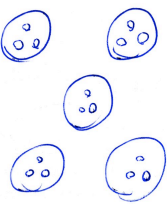
In scoring responses to the task, the teachers were allocated one point for correctly matching all three elements in the row in part i, another point for matching all three elements in part ii (while describing the other form of division), and another for correctly identifying quotient in part d) and providing an appropriate explanation for this. This gave a total of three points if all parts were answered satisfactorily. Two researchers (one of the authors and a PhD student) coded all items independently, with 96% agreement; a very high level of inter-rater reliability.

Examples of satisfactory responses are provided in Table 1. The first row and second row could be swapped. It was not essential that the teacher identify partition first, though this was most often the case (76%).

Table 2 shows the percentage of teachers in Prep to 2 and Year 3 to 4 who represented satisfactorily *each* form of division (by naming it, providing a matching representation and a matching word problem) and the percentage of teachers who named quotient as the model that is most likely to help make sense of $8 \div 0.5$ with an appropriate explanation.

Results indicated that, in relation to the prompt, Prep to 2 and Year 3 to 4 teachers were about twice as likely to represent partitive division successfully than quotientive division. Year 3 to 4 teachers were about twice as likely to make sense of $8 \div 0.5$ by connecting it to an understanding of quotientive division than Prep to 2 teachers.

Table 1
Examples of Satisfactory Responses for Division Stories

Name the form of division	Draw a simple picture to represent this form of division	Write a story problem that represents this form of division for the problem $15 \div 3$.
Partition (shared between; sharing/ed; repeated subtraction)		There were 15 lollies shared between 3 children. How many lollies did each child get?
Quotition (How many; groups of; groups; grouping)		There were 15 children. The teacher put them into groups of three. How many groups were there?

d) Quotition division is selected.

e) An explanation of the kind: “Because it is easier to think of how many halves than sharing between half” or an explanation that indicates the result of the equation is 16 and includes a drawing or description of how this was obtained. For example “each of the 8 is cut in half.”

Table 2
Frequency and (Percentage) of Teachers Giving Satisfactory Responses to Each of the Three Parts of the Division Stories Item.

	Partition	Quotition	$8 \div 0.5$
P - 2	23	12	9
<i>n</i> = 51	(45.1%)	(23.5%)	(17.6%)
3 - 4	24	11	14
<i>n</i> = 41	(58.5%)	(26.8%)	(34.1%)
P - 4	47	23	23
<i>n</i> = 92	(51.1%)	(25.0%)	(25.0%)

It is interesting to note that of the entire group of 92 teachers, 23 represented correctly quotitive division and 23 teachers made sense of $8 \div 0.5$, however only 13 teachers were common to these two groups. This means that 10 teachers could describe quotition but not transfer this understanding to division with a decimal less than one. On the other hand, 10 teachers were able to make sense of $8 \div 0.5$, by connecting it to the process of quotition division while not satisfactorily describing quotition division. The findings of Tirosh and Graeber (1990) that “the majority of preservice teachers agree with the explicit statement “in a division problem, the quotient must be less than the dividend” (p. 99) is of little surprise in light of these data.

It may be fair to say that while 75% of teachers had difficulty making sense of $8 \div 0.5$ in a way that would be helpful to model for students, we were assessing this understanding of a group of teachers who may not focus on this in their year level. The authors will be

interested to see if teachers in Year 5 to 6 are more likely to have a satisfactory explanation when this item is administered in 2009.

While the authors attempts with this item are to assess teachers' PCK, it is sometimes a fine line between what is PCK and what Chick (2007) described as content knowledge in a pedagogical context. Some teachers revealed difficulty with the mathematics of the expression ($8 \div 0.5$). Examples included:

If I was teaching this it is easier to start with 8 and halve it.

You have the total amount and it is shared between 2, therefore 4.

Because you are splitting 8 in $\frac{1}{2}$.

Share 8 between 2.

Reponses such as these indicate that the teachers considered incorrectly that $8 \div 0.5$ could be thought of as half of 8 or $8 \div 2$. Also 27 teachers out of the 92 left this part of the question blank.

It could be argued that the scoring scheme could be penalising teachers unfairly who were able to correctly provide a word problem and a matching drawing but were unable to give some name to this form of division. Table 3 indicates the percentages of correct responses for the three parts of this item if we remove the need to "name the form of division," with the additional numbers in each category shown in square brackets.

Table 3
Frequency and (Percentage) Satisfactory Responses if Naming Not Required.

	Partition	Quotition	$8 \div 0.5$
P - 2	29	15	12
$n = 51$	(56.9%)	(29.4%)	(23.5%)
	[6]	[3]	[3]
3 - 4	31	13	15
$n = 41$	(75.6%)	(31.7%)	(36.6%)
	[7]	[2]	[1]
P - 4	60	28	27
$n = 92$	(65.2%)	(30.4%)	(29.3%)
	[13]	[5]	[4]

Discussion

In relation to teachers' performance on the item discussed in this paper, few experienced primary mathematics teacher educators will be surprised that the item proved difficult. However, the extent of the difficulty may surprise, and also particularly the lack of facility with quotitive division. If the content is regarded as important for primary teachers to know, and this appears to be accepted in the mathematics education community, then careful thought needs to be given to how best to present it to teachers.

In 2009, the authors have been more explicit with teachers about the two forms, have taken time to ask them to model each with materials and pictures, and to consider a range of story problems in relation to which form of division might more easily enable problem solutions for each. The authors also agree with Mulligan and Mitchelmore (1997) that greater attention could be given to the presentation of word problems involving rational numbers earlier in the school curriculum, with an emphasis on developing appropriate intuitive models to assist in understanding and solving them. As Tirosh (2006) noted, "the predominance of the primitive, partitive model has been shown to seriously limit children's

and prospective teachers' abilities to correctly respond to division word problems involving fractions" (p. 7).

In teacher education settings, it is also important to negotiate meaning for symbols for division, so that teachers can do this with their students. Anghileri (1995) noted that there are many common phrases used by teachers in interpreting $12 \div 3$, including "divided by 3, 12 divided into 3, 12 shared by 3, 12 shared into 3, and how many 3s go into 12?" (p. 12). It will be important for teachers to emphasise that different language is appropriate to different contexts and situations, while recognising that "the advantages of exploiting vocabulary identified with everyday experiences must be set against the underlying primitive and limiting model that sharing will provide for division in later mathematics" (p. 13).

Looking at the broader issue of assessing PCK in mathematics, although it is early days in our work with teachers in this project, several important findings have emerged. Although clearly not as extensive as the framework of Chick (2007) and others, if teachers can demonstrate capacities in the areas of Pathways, Selecting, Interpreting, Demand and Adapting, as we have defined them, it can be argued reasonably that they are likely to have the PCK to teach mathematics well. There is always a tension between collecting large amounts of high quality data from a small number of teachers and collecting less data from a larger number of teachers. A balance has been struck which works well for the research and requirements of the funding bodies.

Multiple-choice items, without the opportunity for teachers to elaborate their decision making, are not very useful (see, e.g., Clements & Ellerton, 1995). In the authors' opinion, a major weakness of much of the work of Ball and her colleagues (e.g., Ball & Bass, 2000) is that items take this form. Multiple-choice was used in 2008 for some items, but it was realised that they had the same weakness as many state and national tests, namely that if 27% of respondents choose the right alternative from five choices, there is no way of knowing how many guessed, and the reasons for a particular choice, unless this is sought as part of the response. In 2009, the intention is that any multiple-choice items will require teachers to justify their choices.

The tasks which have developed have the capacity to demonstrate growth over time in PCK. Although space did not allow the authors to provide detailed data on this, the average score on the PCK questionnaires increased from 5.1 to 6.1 (out of 11), 7.6 to 10.0 (out of 17); and 7.6 to 10.8 (out of 18), for Prep to 2, Year 3 to 4, and Year 5 to 6 teachers, respectively. This result is encouraging in relation to future work in the project.

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