# Linear Algebra Snapshots through APOS and Embodied, Symbolic and Formal Worlds of Mathematical Thinking

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Linear algebra is one of the unavoidable advanced courses that many mathematics students encounter at university level. The research reported here is as part of the first named author's recent PhD thesis where she created and applied a theoretical framework combining the strengths of two major mathematics education theories in order to investigate the learning and teaching of some linear algebra concepts. This paper highlights some of the overall findings of this research and suggests applications for learning and teaching in undergraduate mathematics classrooms.

## Introduction

In recent years many mathematics education researchers have been concerned with students' difficulties related to the undergraduate linear algebra courses. There is agreement that teaching and learning this course is a frustrating experience for both teachers and students, and despite all the efforts to improve the curriculum the learning of linear algebra remains challenging for many students (Dorier & Sierpinska, 2001). Students may cope with the procedural aspects of the course, solving linear systems and manipulating matrices, but struggle to understand the crucial conceptual ideas underpinning them. The concepts may be presented through a definition in natural language, which may be linked to a symbolic presentation. These definitions are considered to be fundamental as a starting point for concept formation and deductive reasoning in advanced mathematics (Vinner, 1991; Zaslavsky & Shir, 2005). Interestingly enough, at the end of the course many students do reasonably well in their final examinations, since the questions are mainly set on using techniques and following certain procedures, rather than understanding the concepts (Dorier, 1990). This, as Sierpinska, et al. (2002, p. 2) describe is a "waste of students intellectual possibilities". They believe "linear algebra, with its axiomatic definitions of vector space and linear transformation, is a highly theoretical knowledge, and its learning cannot be reduced to practicing and mastering a set of computational procedures" (*ibid,* p. 1).

The action-process-object-schema (APOS) development in learning proposed by Dubinsky and others (Dubinsky & McDonald, 2001) suggests an approach different from the definition-theorem-proof that often characterises university courses. Instead mathematical concepts are described in terms of a genetic decomposition into their constituent actions, process and objects in the order these should be experienced by the learner. In more recent years Tall has introduced the idea of three worlds of mathematics, the embodied, symbolic and formal (Tall, 2004). The worlds describe a hierarchy of qualitatively different ways of thinking that individuals develop as new conceptions are compressed into more thinkable concepts (Tall & Mejia-Ramos, 2006). The embodied world, containing embodied objects (Gray & Tall, 2001), is where we think about the things around us in the physical world, and it "includes not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuo-spatial imagery." (Tall, 2004, p. 30). The symbolic world is the world of procepts, where actions, processes and their corresponding objects are realized and symbolized. The formal world of thinking

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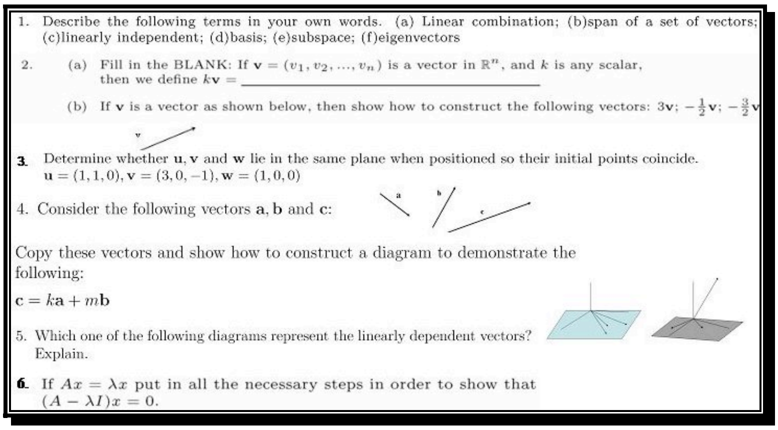
comprises defined objects (Tall, Thomas, Davis, Gray, & Simpson, 2000), presented in terms of their properties, with new properties deduced from objects by formal proof. To examine the usefulness of these theories in learning and teaching, for each chosen linear algebra concept in the research (vector and scalar, linear combination, linear Independence/dependence, span, basis, eigenvalues and eigenvectors) a preliminary framework was constructed (see Figure 1 for the framework for vector and scalar multiplication). The framework was constructed by creating a grid with 12 cells to examine a learner's action, process, and object thought processes of the concept (the left-hand column) in each of the three mathematical worlds of embodied, symbolic and formal (the top cells). This formulation was achievable since it is possible that students can do actions, think about processes, and encapsulate their processes to form objects in each of the embodied, symbolic and formal worlds of mathematical thinking. Despite the fact that Dubinsky's APOS theory refers to learners' mental views and Tall's worlds are about mathematical thinking, the theories seem to blend naturally together. Such a framework allows researchers to evaluate students' conceptual understanding of linear algebra and observe the way students learn. Furthermore, it was designed to help teachers and instructors to cover a spectrum of representations in the classroom in such a way that teaching based on it would help students build linear algebra knowledge and give them the impression that mathematics is not "completely cut, dried and salted away" (Mason, 2002, p. 4). The goal of this research was to find the level of students' conceptual and procedural understanding of the linear algebra concepts, their difficulties with these concepts, the effects of using embodied ideas in teaching/learning of linear algebra, and the usefulness of the framework.

## Method

This research comprised several qualitative case studies to study students' thinking about some basic linear algebra concepts, namely vector, scalar multiple, linear combinations, linear dependence/independence, span of vectors, subspace, basis, and eigenvectors and eigenvalues. The participants were first and second year general mathematics students from the University of Auckland who had volunteered to take part in this study. The lectures were taught by first named author and were designed based on the proposed framework (see example in Figure 1) to give students the overall experience of the concepts in the embodied, symbolic and formal worlds of mathematics. For example, linear independence of vectors was presented by showing embodied, visual aspects of the concept first. This was then linked to the notion of linear combinations in the form of algebraic and matrix symbolisations. The formal definition was given after the symbolic and visual aspects were addressed. Students were given a set of questions (see sample in Figure 2) on variety of concepts in linear algebra, which was designed to examine their embodied, symbolic and formal understanding, rather than the procedural abilities. As part of the case studies interviews and concept maps were also employed. For comparison reasons a recent PhD in mathematics also participated in this study. To protect students' privacy students are referred by the case studies that they were involved in and a number as they were listed. For example student 2B-5 refers to the fifth second year student on the list from case study 2(b). For further details see the first-named author's thesis (Stewart, 2008).

| Worlds / APOS | Embodied World | Symbolic World: Algebraic Rep. | Symbolic World: Matrix Rep. | Formal World |
|---|---|---|---|---|
| **Action** | Can see vector as displacement from $A$ to $B$<br>Can perform a scalar multiplication ($\mathbf{a}$, $2\mathbf{a}$) | Can multiply a vector by a scalar e.g $3\mathbf{a}$ | Can add vectors $\begin{bmatrix}2\\0\\1\end{bmatrix}+\begin{bmatrix}4\\3\\0\end{bmatrix}=\begin{bmatrix}6\\3\\1\end{bmatrix}$<br>Can multiply a vector by a scalar $2\begin{bmatrix}4\\3\\1\end{bmatrix}=\begin{bmatrix}8\\6\\2\end{bmatrix}$ | |
| **Process** | Can recognise equivalent vectors represented by parallel arrows, having same length and direction<br>Can add<br>Can perform scalar multiplication for a general case ($\mathbf{a}$, $k\mathbf{a}$; ma, a, ka) | Can understand vector addition/subtraction scalar multiplication parallelogram/triangle rules $k\mathbf{v}\forall k \in \mathbb{R}$ $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$ $\forall \mathbf{v},\mathbf{w} \in V$ $\mathbf{v}=\mathbf{w}$ equivalent vectors | Vector addition/subtraction scalar multiplication, the vectors $\mathbf{u}=\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$ are said to be equal if $x_1=x_2$, $y_1=y_2$, $z_1=z_2$ | |
| **Object** | Can see vector as a directed line segment with magnitude, which can be picked up mentally and moved around (a free vector) | Can see that the vector $\mathbf{v}$ can be treated as an entity and operated upon e.g. $f(v): \mathbf{v_1} \to \mathbf{v_2}$ | The column vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ or row vector $(x, y, z)$ can be treated as an entity and operated upon e.g. $f\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)$ | The $n$-tuple $\begin{bmatrix}a_1\\a_2\\\vdots\\a_n\end{bmatrix}$ as an element of $R^n$ is a vector, which holds all the properties of a vector space. An element of a vector space $\mathbf{V}$, can be operated on. e.g. $T: v_1 \to v_2$ $T(v) = A.v$ Can understand the definition of scalar product over any field $F$ |

*Figure 1.* A framework for the concept of vector and scalar multiplication.


1. Describe the following terms in your own words. (a) Linear combination; (b)span of a set of vectors; (c)linearly independent; (d)basis; (e)subspace; (f)eigenvectors

2. (a) Fill in the BLANK: If $\mathbf{v} = (v_1, v_2, ..., v_n)$ is a vector in $\mathbb{R}^n$, and $k$ is any scalar, then we define $k\mathbf{v}$ = ____________

   (b) If $\mathbf{v}$ is a vector as shown below, then show how to construct the following vectors: $3\mathbf{v}$; $-\frac{1}{2}\mathbf{v}$; $-\frac{3}{2}\mathbf{v}$

3. Determine whether $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ lie in the same plane when positioned so their initial points coincide. $\mathbf{u} = (1, 1, 0), \mathbf{v} = (3, 0, -1), \mathbf{w} = (1, 0, 0)$

4. Consider the following vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$:

Copy these vectors and show how to construct a diagram to demonstrate the following:

$\mathbf{c} = k\mathbf{a} + m\mathbf{b}$

5. Which one of the following diagrams represent the linearly dependent vectors? Explain.

6. If $Ax = \lambda x$ put in all the necessary steps in order to show that $(A - \lambda I)x = 0$.

*Figure 2.* A sample of test questions.

# Results

The analysis of the data is based on the progression of the featuring concepts categorised into the following groups: vector and scalar multiples; linear combinations and span of vectors; linear dependence/independence of vectors; basis and subspace; eigenvalues and eigenvectors. Two reasons behind the selection in this study were firstly that they occur early in linear algebra courses, and secondly, they represent a natural progression of ideas from vector and scalar multiple to the study of eigenvalues and eigenvectors. For many students the rapid development of these concepts, sometimes within a single lecture, creates problems as they are built on each other and soon are considered as assumed knowledge. It was hoped that investigating each group of concepts in detail may assist in understanding the reasons behind these difficulties. The discussion below gives a brief description based on the overall findings.

### *Students' Thinking About Vector and Scalar Multiplications*

The analysis of the questions on vector and scalar multiples showed that although students were able to do trivial actions, for example adding two vectors geometrically, they had difficulty in seeing a vector as an object (free vector), and thus were unable to add vectors geometrically, which requires one to be lifted up and then added to the tip of another vector. The symbolic-process view of scalar multiple, $kv$, of a vector $v$ was also difficult for some students as they did not consider $k$ as a scalar and instead thought of $k$ as a vector, thus they performed a scalar product (or a dot product) between a scalar and a vector. This possibly shows their confusion between the two notions of scalar multiple and scalar product (a language difficulty) and also confirms their unfamiliarity with recognising a vector and a scalar in different representations (in this case symbolic-algebraic). The embodied ideas of scalar multiple of a vector were mainly perceived procedurally by many students who did not have a process view of scalar multiples of the given vector. Consequently, most students only drew several separate lines (see Figure 3) and did not consider showing all the scalar multiples on a single straight line (a process).

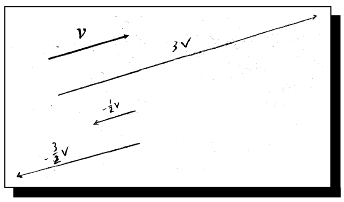


*Figure 3.* Student 1B-8's response relating to the scalar multiplication of the vector.

### *Students' Thinking About Linear Combination and Span of Vectors*

Students' responses to questions on the concept of linear combination showed that a majority of first year students had considerable difficulties with defining the concept and the notion seemed foreign to most students. Some thought of 'linear equations' instead of 'linear combinations', with one student even giving the equation of a line $y = ax + b$, possibly due to a language difficulty. Evidence showed that most second year students who were about to start the course had absolutely no idea about this concept from their previous courses. Also based on the findings in this research, right at the end of the semester some very successful second year students struggled to remember the key aspects of the concept, as this concept was often introduced through a formal definition. For example in response to a request to define the term linear combination, 45.5% of students in case study 2(b) ($N$=11) did not write any answer, and the remaining 54.5% only gave procedural or incomplete responses. Student 2B-2's response in defining the term linear combination was: "something like $xv + yu$; $x, y$ belong to $\mathbb{R}$". In an interview he said "linear combination, hmm . . . I can't quite remember the definition, I can just remember those forms something like $b = x_1v_1 + x_2v_2$ and something like that and $x$ belong to $\mathbb{R}$. I only can remember these things". When he was asked for further explanation he said: "Hmm difficult! Linear combination is an object class in a space formed by the two vectors and *x*, *y* are scalars, this is my understanding of linear combination". This clearly demonstrates his lack of knowing the definition and not having an object view of the concept in general.

In contrast, building the concept through the embodied, symbolic and formal worlds of mathematics, starting from vectors and scalar multiples seemed to be effective for those who attended the researcher's summer course in case study 2(c) (N=16), since at the end of the first lecture, 72.5% of students were able to give a description (mainly at the action or process level with occasional ones at the object level) for the concept in their own words. It was also noted that a majority of students had the procedural abilities to calculate algebraically the linear combination (actions), however, some first and second year students were affected by their poor arithmetic skills. Student responses to Question 4 relating to the embodied aspects of vector addition and linear combination showed misconceptions in a number of areas which mainly rose from their unfamiliarity with the concept of vector. Only 37% of first year students (in the first test), and 45.4% of second year students from case study 2(b) drew the correct diagram (a process) as some students lacked basic skills constructing parallelograms or triangles, keeping the correct direction for the given vectors, and constructing the correct direction for the resultant vector. However, a high percentage (68%) of the researcher's students from case study 2(c) were able to draw a correct (an embodied process) diagram (See figure 4). On the other hand, the PhD graduate who gave a object-formal definition for the term linear combination, had no problem with symbolic-algebraic manipulations of vectors and was able to recognise the symbolic-algebraic representation of the linear combination and draw an embodied-process representation.

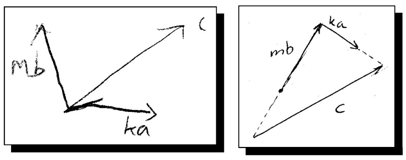



*Figure 4*. Students 2B-1 (LHS) and 2C-1's embodied thinking of linear combination.

### *Students thinking about linear independence/dependence*

Students' responses to the concept of linear independence showed that although students from case study 2(b) struggled to define clearly the term linear independence, most students from the researcher's class were confident describing the concept in their own words. In Question 3, to examine students symbolic-matrix action views many students from both groups (case studies 2(b) and 2(c)) created a matrix and some showed a symbolic-matrix view, and two students from case study 2(b) and eight from case study 2(c) concluded that the vectors were linearly independent (a process-symbolic), although this concept was not an explicit requirement for this question. Moreover, five students from the researcher's class also approached this question visually by drawing the vectors on a diagram (versatile thinking), showing the ability to link representations (see Figure 5). In Question 5, related to distinguishing between the two diagrams (an embodied-process view), only 36% of students from case study 2(b) were able to relate the correct diagram to the concept of linear dependence. However, all students in the researcher's class choose the correct diagram.

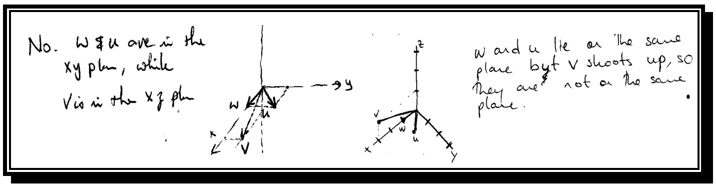



*Figure* 5. Students 2C-1 (LHS) and 2C-2's use of diagrams as part of their embodied world thinking.

### *Students' Thinking About Basis and Subspace*

In order to construct the concept of basis students need to build on a number of previous concepts. A genetic decomposition (GD) (e.g. Czarnocha, Loch, Prabhu, & Vidakovic, 2001) of basis requires a combination of GD's of span and linear independence of vectors, a link that the students in this study often didn't make. The majority of students in case study 2(b) were unable to define the terms basis and subspace, with eight students writing nothing at all. Those who gave a definition for the term basis, gave a procedural definition based on an action relating to find a basis (action-symbolic-matrix view). This is not surprising since it is easier than grappling with the formal world ideas, and is the method emphasised in the course. For example, the course notes speak about how *to find* "a basis for the Nullspace of an $m\times n$ matrix *A*", "a basis for the column space of a matrix", and "the span of a set of vectors $v_1$, $v_2$,…$v_n$, [by forming] the matrix $A$ = [$v_1$, $v_2$,…$v_n$] with these vectors as its columns", and gives a symbolic, matrix method for each. In contrast, only three students in the researcher's class did not give a definition, the majority of students were able to mention that the vectors must be linearly independent and span, which was also demonstrated in students' concept maps (see Figure 6).

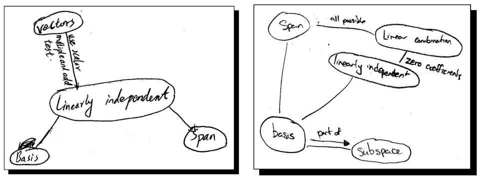



*Figure* 6. Student 2B-2's (LHS) and student 2C-5's linking of basis to span and linearly independent is shown in their concept maps.

### *Students thinking about eigenvectors and eigenvalues*

The majority of students were unfamiliar with embodied ideas of eigenvalues and eigenvectors, and they preferred an action-symbolic approach in describing the term eigenvector. For example in case study 2(a) (N=42) a majority of students referred to the procedural part of the definition ($Ax = \lambda x$). Results also indicated the first year students' difficulties in case study 1(a) (N=10) in finding the eigenvectors and eigenvalues (action-symbolic-matrix manipulations). Question 6 (a process-symbolic-algebra question) was designed to discover a possible process-object complexity with $Ax = \lambda x$, in the sense that although the two sides of the equation are quite different processes, they have to be encapsulated to give equivalent mathematical objects. In this case the left hand side is the process of multiplying (on the left) a vector (or matrix) by a matrix, while the right hand side is the process of multiplying a vector by a scalar resulting in the same vector as the final object. Results showed that the progression, working within the algebraic symbolic world, from $Ax = \lambda x$ to $(A - \lambda I)x = 0$ is not perceived as straightforward by many students as they have difficulty by the two different processes in the first equation, and do not know what identity the *I* refers to. Figure 7 shows that this affected students' ability to complete the relatively simple three-line transformation of the equations.

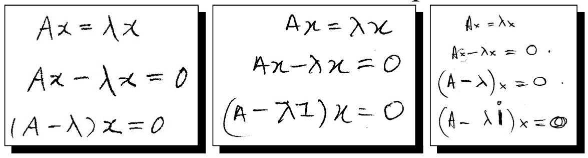


*Figure* 7. Working of students 2A-41, 2A-24 and 2A-34 on question 6 (left to right).

## Discussion and Conclusions

The extensive evidence revealed that the majority of students had major problems understanding the concepts that are the essence and foundation of a linear algebra course. Many students had difficulties connecting their understanding of one concept to other related concepts (a process-formal). For example many students were not able to link basis to linear independence and span. It seems that a sizable number of students had difficulty in expressing their understanding of the definitions. The framework proved to be a valuable tool in analysing students' thinking by providing evidence of students' level of thinking based on the specific cells or regions of the framework. For example we could use the framework to trace where students' thinking was at. By finding out a student's weak points in their understanding and thinking, the instructor can see the areas that need improvement and how to address them. Based on the results of this research students were mainly thinking and representing their understanding of the concepts in a manner described by the action-symbolic-matrix or/and process-symbolic-matrix cells of the framework (the four cells in the centre section). Although a number of students from the first-named author's class demonstrated embodied or object views, the majority of students tended to show an action/process view in the symbolic world. So the question would be, if the three worlds of the mathematical thinking are hierarchical, how did the students reach the symbolic world without passing through the embodied? In other words if student thinking is based in the symbolic world surely they would have had embodied ideas too, since they are relatively easier than the symbolic ones. The answer to this question is not trivial. In Tall's description of the three worlds, he often refers to the entire mathematics from school mathematics to calculus and more advanced algebra right through to the definitions and axioms in the formal world. There has been no study examining the development of a single concept of advanced mathematics through these three worlds. How could we apply his theory to a single concept? In other words, to construct conceptual understanding does one have to start from the embodied, travel through the symbolic, and finally arrive at the formal world? As Tall reveals in an *ideal* world this is likely the case. Most students need to symbolise the embodiment and embody the symbolism first and only after fully integrating them they will reach the formal world. However, in the *real* world it is possible to be solely in the symbolic world of thinking by following the steps of the instructor in the class. In contrast, a mathematician can comfortably live in the symbolic and embodied worlds since he is able to reverse and construct embodied views, as well as going forward to the formal world (as we observed in the case of the PhD graduate in this study). The possession of a rich schema allows him/her to tie all the pieces of his knowledge in a way that the student may not be able to. Thus, the claim is that it is the embodied view that gives deep meaning to the concept allowing us move toward the formal world. In the case of the students in this research, it appears that since they often lacked embodied aspects of the concepts and were trapped in the symbolic world, and were not able to move to the formal world of mathematical thinking. It is suggested that a central goal of mathematics education should be to increase the power of students' representations (Greer & Harel, 1998). Employing a visual, embodied approach to the teaching of linear algebra concepts, which are often treated symbolically or formally, may enrich students' understanding and satisfy this goal (Stewart & Thomas, 2007).

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