Multimodal Use of Semiotic Resources in the Construction of Antiderivative

Michael O. J. Thomas — *The University of Auckland, NZ* — <moj.thomas@auckland.ac.nz>

Caroline Yoon — *The University of Auckland, NZ* — <c.yoon@auckland.ac.nz>

Tommy Dreyfus — *Tel Aviv University, Israel* — <tommyd@post.tau.ac.il>

Mathematical knowledge production is increasingly seen as a multimodal activity, involving complex synchronic and diachronic interactions between semiotic systems. This paper documents a diachronic semiotic analysis of interactions that occurred as two secondary teachers engaged in a mathematising task requiring them to construct graphically an antiderivative, using the context of tramping. The roles of a number of different semiotic resources that were prominent at various times are described, along with suggestions on possible cognitive advantages these offered in terms of reduced cognitive load and increased affordances.

Background

A major part of the complexity of mathematics is the multitude and diversity of semiotic signs and their relationships embodied in the subject. In this paper the semiotic analysis is based on the triadic model of Peirce, where, in his semiotic triangle, physical, material signs are made sense of through thoughts and ideas, which bring to mind an object (the referent). While Peirce categorises signs into icons (or likenesses), indexes, and symbols, which are not mutually exclusive, nothing is automatically a sign; it has to be interpreted as a particular sign by an individual. Symbols are signs that have become associated with their meaning by accepted usage (Peirce, MS404, 1894); they become significant simply by virtue of the fact that they will be so interpreted, rather than being directly related by resemblance or physical connection to the referent. In mathematics signs are usually grouped together into families, which include logic signs, algebraic symbols, matrices, and graphs, which have been called semiotic systems or semiotic registers (Duval, 2006). It is the association of the sign with a semiotic system that gives its specific interpretation. In a mathematics education context Kaput (1989) has referred to these registers as *representation systems*, and in this paper we will follow this terminology. We will also denote a sign associated with a given representation system as a representation (see Thomas, 2008).

In their analysis of mathematical learning Arzarello, Paola, Robutti, and Sabena (2009) employ a wider than usual definition of the term sign, including for example, spoken words and gestures (we could also add sounds, odours and other signs here). They use the term *semiotic resource* for such a sign, enabling us to speak, for example, of a representation system of gestures. However, representations are passive entities in terms of learning, even if they take a dynamic form. It is the intention or attention of the learner (or teacher), through the process of semiosis, that transforms them into an active state, into a semiotic resource. Radford (2009) also espouses this position, proposing that thinking does not occur solely in the head but also in, and through *sensuous cognition*, a sophisticated semiotic coordination of speech, body, gestures, symbols and tools. Of course, representations are not immutable but can be transformed by the individual. Duval (1999, 2006) has delineated two categories of such representational transformations; those within the same register—*treatments*—and those that change a register without changing the object—*conversions*.

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Important aspects of understanding a mathematical concept arise from the ability to relate, through conversions, corresponding elements of different representations from different representation systems that signify the concept. Confirming their importance Duval (2006, p. 116) writes "If one wishes to analyze difficulties in learning mathematics it is to the study of conversion of representation that one ought to give priority and not that of treatments." Thomas (2008) includes this crucial aspect of conversion between representation systems, along with treatments, as one aspect of versatile thinking, defining "*representational versatility*—the ability to work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations." (*ibid*, p. 10). It is active engagement that makes the representation a *cognitive tool* according to Thomas (2008), turning it into a semiotic resource, and changing it from a passive or inert structural entity with no intrinsic meaning to an active construct.

Mathematical learning and teaching are increasingly seen as multimodal processes (Radford, 2009; Arzarello, 2006) requiring orchestration of a variety of semiotic resources. Arzarello (2006) claims that, whatever the interactions, a narrow focus on a single representation system is insufficient to capture the complexity of multimodal learning in the mathematics classroom. Hence he has described (Arzarello & Robutti, 2008) the concept of *semiotic set* as an extension of semiotic system, comprising signs, their modes of production and transformation, and the relationships among the signs and their meanings. In turn a *semiotic bundle* is defined as a collection of semiotic sets along with the mutual relationships between the sets of the bundle (Arzarello & Robutti, 2008). It is this semiotic bundle that gives rise to multimodal thinking (Arzarello et al., 2009). Arzarello (2008, p. 162) also introduces a space in which the semiotic bundle exists, and where "different perspectives can be combined in a shared environment for cognition. I call it the cognitive space of action, production and communication (APC space)", noting that "gestures constitute an important ingredient of learning, hence of the APC space" (*ibid*, p. 170). The implication is that gestures may be viewed as cognitive rather than physical entities. Thus the APC space is an integrated system comprising different modalities such as written and oral language, symbols, gestures, etc., and it is the multimodal interactions among the different registers or representations systems that are at the core of mathematical knowledge production (Arzarello, 2006). Analysis of these interactions can be synchronic, investigating those occurring at the same time, or diachronic, looking at how interactions develop over time. Both are important to appreciate how the semiotic bundle is instrumental in the construction of learning.

In this paper we use the framework of semiotic bundle to undertake a primarily diachronic analysis of the multimodal use of semiotic resources in the graphical construction of aspects of the antiderivative concept. As part of this analysis we consider the qualitative nature of representational transformations, along with interpersonal and representational interactions.

Method

The research employed a case study design in which two female secondary school mathematics teachers—Ava, with seven years of teaching experience, and Noa, who had three years of teaching experience—worked together on a sequence of four related, primarily graph-based, activities designed to assist constructing the notion and properties of antiderivatives. While they worked a researcher was present, who tried to remain neutral in the process but responded to questions (without leading the participants) and provided encouragement when required. Each activity lasted for approximately one hour, during which time the teachers were videotaped and audiotaped. They knew each other well, but neither had taught calculus at Year 13 level (the last year of secondary school in New Zealand). In this paper we focus solely on the main task of the first activity (see Figure 1), to produce, in the context of a tramping expedition, the distance-height graph of a track from its gradient graph. Following a few readiness and warm-up tasks, in which they were given a distance-height graph of a tramping track, and asked to find its gradient graph, the main question was introduced.

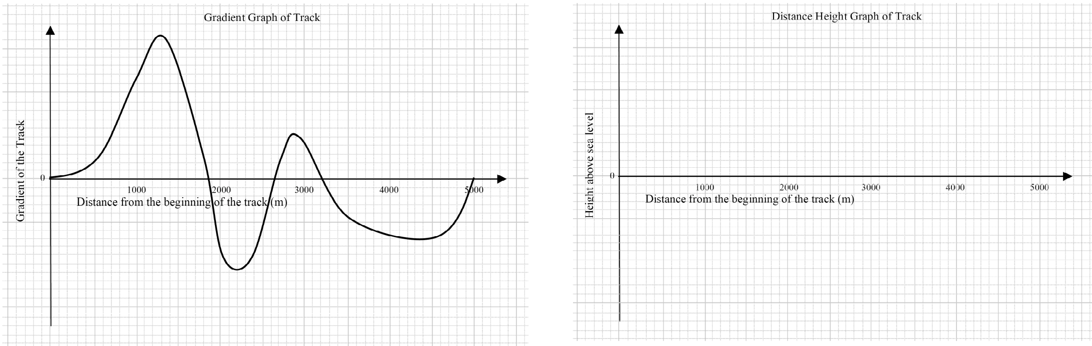

Figure 1. The tramping track task (the second graph was presented directly below the first).

Results and Analysis

As the teachers worked to construct the antiderivative graph their semiotic bundle changed, with different semiotic resources coming to the fore in the influence on their thinking. Hence a number of diachronic snapshots across the task activity produce a semiotic bundle that appears qualitatively different at each stage, and four of these are summarised in Figure 2.

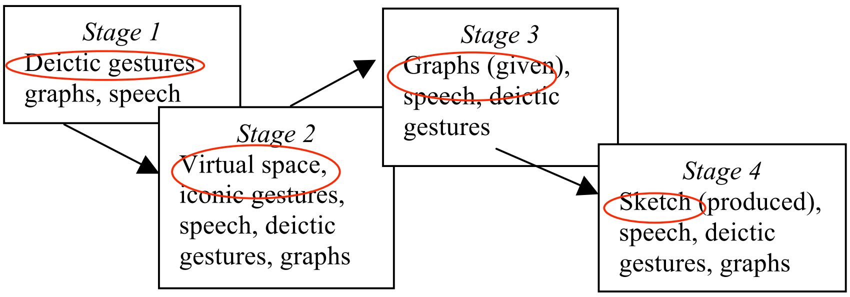

Figure 2. A summary of components of four stages of the semiotic bundle.

As Figure 2 shows, in each of these stages the semiotic bundle comprises a number of semiotic resources. The semiotic resources within the bundle at any one time are active concurrently in the APC and are interlinked. For example, within the semiotic bundle in stage 2, the teachers point to the gradient graph (using deictic gestures), while describing the gradient at each region they are pointing to (using speech), and drawing the corresponding graph of the track (using the graphical representation), while sometimes using their hands to act out complex parts of the track in virtual space (using iconic gestures and the virtual space) (Yoon, Thomas, & Dreyfus, 2009). In this paper, we track the changes in the semiotic bundle that occur as it transitions from one stage to the next, in terms of the key semiotic resource in the forefront of the teachers' thinking. In this manner we can explore effectively how new semiotic resources are created and used in time. Thus, we will primarily focus on one semiotic resource at each stage of the bundle, and analyse how it gave rise to the next, and the progression examined is shown in Figure 3. While a synchronic analysis is not our primary focus of attention here, we relate to it too, where this assists in illuminating epistemological progress as construction of the antiderivative proceeds.

Deictic gestures → Virtual space → Graph → Sketch

Figure 3. The progression of foreground semiotic resources in the semiotic bundle.

Semiotic Bundle Stage 1

At this stage the semiotic resource of deictic gesture appears to be cognitively to the fore. Both Ava and Noa use such gestures in a process interaction (considering sections of the graph but traversing them pointwise) with the graphical representation to build an embodied perspective of the track from the gradient. They trace with their finger/pen along the gradient graph (see Figure 4), while verbalising the uphill/downhill portions of the track.

133	Ava	So you are going uphill *[traces along 1a]* downhill [*traces along 1b*].
134	Noa	Get to the top *[traces along 1a],* downhill *[traces along 1b].*
135	Ava	Up a little bit *[traces along 1c]*, downhill *[traces along 1d]*.

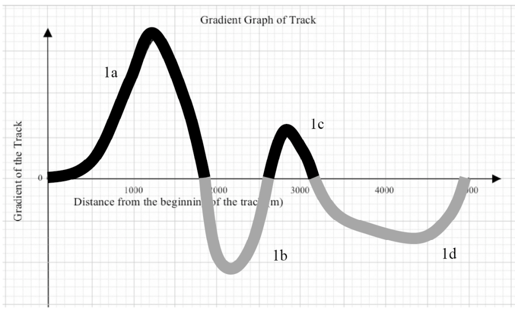

Figure 4. Sections of the gradient graph traced with deictic gestures.

Next, they refine this by considering the relative magnitude of the gradient over section 1a – i.e., that some parts of the track will be steeper than others. Once again they construct this understanding in embodied terms, in the context of the difficulty of walking the track – "this is difficult walking", "that's the hardest place". They use deictic gestures to clarify a disagreement about the gradient getting less steep at the decreasing portion of 1a. They use tracing and pointing for different areas – Ava emphasising the motion of the decreasing gradient "getting gentler" by tracing, whereas Noa emphasises the relative height of the gradient graph "it's still hard, still high gradient" by pointing to the region. Here, the deictic gestures play a crucial role in communicating different concepts – the *way* the teachers point to or trace part of the graph is at least as important as the section of the graph they gesture towards. Although they use deictic gestures to trace along the gradient graph in subsequent stages of the semiotic bundle, they do not pay as much attention to the actual gradient graph as they do in this first stage. Since the gradient graph semiotic resource is not linked to many other semiotic resources, this enables them to concentrate more closely on the regions in the given gradient graph to which they are pointing.

In this first stage, they focused on some of the most salient mathematical features of the gradient graph. First they recognised that it could be divided into mutually exclusive sections where the gradient was positive or negative, and that these corresponded to uphill and downhill sections on the antiderivative track. Second, for each section they considered the change in the gradient of the track, as indicated by changes in the height of the gradient graph, and related this to the slope of the antiderivative track.

Semiotic Bundle Stage 2

The next semiotic resource that takes on a primary role in the semiotic bundle is a cognitive and mathematical virtual space (Yoon, Thomas, & Dreyfus, 2009), which the teachers create through iconic gestures that depict the slope of the track at any given point. They construct an antiderivative representation in this virtual space by again tracing along the gradient graph (see their fingers pointing to the gradient graph on the paper on the table), while simultaneously gesturing the slope of the corresponding part of the track. In the left picture (Figure 5), Noa is gesturing the part of the graph that starts to go up again (1c in Figure 4), and is gesturing what that part of the track looks like. The residues of the slopes that she has gestured form a trail in a virtual space, which Ava points to (Figure 5, right picture), when referring back to a steep part of the track that was previously acted out.

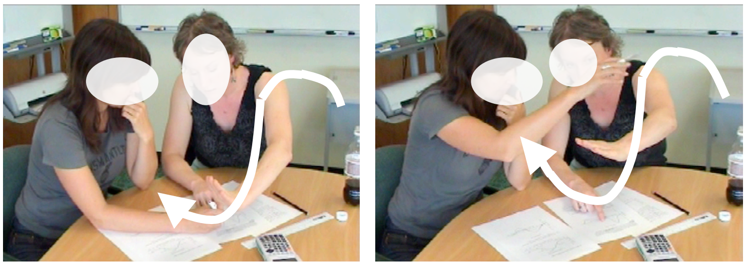

Figure 5. Iconic and deictic gestures used to build the virtual space.

They used the virtual space as a semiotic resource to discuss mathematical ideas about relative steepness, maxima and minima, and constant gradient for the antiderivative, in addition to the positive and negative graph parts described when using only deictic gestures. The mechanics of constructing objects in the virtual space (hand and movement) enabled them to enact the whole antiderivative track using their straightened hands as makeshift tangent lines that showed the changes in slope, and whose trace showed the shape of the graph in the virtual space. Consequently, they could explore questions of subtle changes in gradient in more depth.

Semiotic Bundle Stage 3

The next semiotic resource that came to the fore was a graph of the antiderivative track, which Ava drew by inserting vertical lines from the "important points" on the gradient graph down to a set of *x*-*y* axes directly below (see Figure 6, with her vertical lines enhanced). These "important points" that she chose are the local maxima and minima of the gradient graph, as well as its *x*-axis intercepts. She drew the graph of the track while simultaneously tracing along the relevant portions of the gradient graph, and verbalising what the corresponding parts of the track should look like. While drawing the graph of the track, Ava and Noa discussed for the first time the height of the track at significant places, such as the first "valley" in the track. They discussed whether it occurs at sea level, and whether or not it is possible to tell the height of the track from the graph.

In terms of constructing the mathematical properties of the antiderivative track they now used vertical lines to match up the x-values of key features. This was not possible using the virtual space, because there was no permanent trace left. Hence they had to combine knowledge from their cognitive representation of the virtual space with the given graphical gradient representation. In this way they were able to link: the turning points on the gradient graph with the steepest gradients on the antiderivative; and the zeros on the gradient graph with the turning points on the antiderivative. Further, the permanent existence of the gradient graph, as well as the track axes encouraged them to consider accuracy in representing the shape of the graph. They also needed to consider the height of the graph for the first time, something not constrained by the virtual space. This is seen in the bottom two sketches (Figure 6) where, initially they drew the antiderivative track graph to reach zero height at x = 2600, but later refined this, realising that zero on the gradient graph at this x-value did not necessarily imply that the track reached zero height.

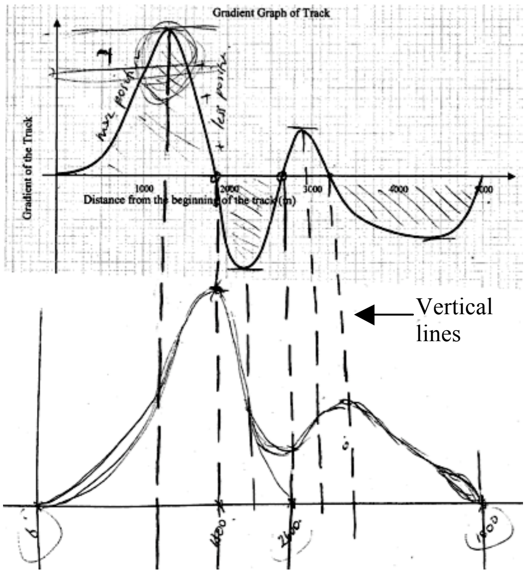

Figure 6. Lines at ‘important points’ build a graphical correspondence relationship at key x-values.

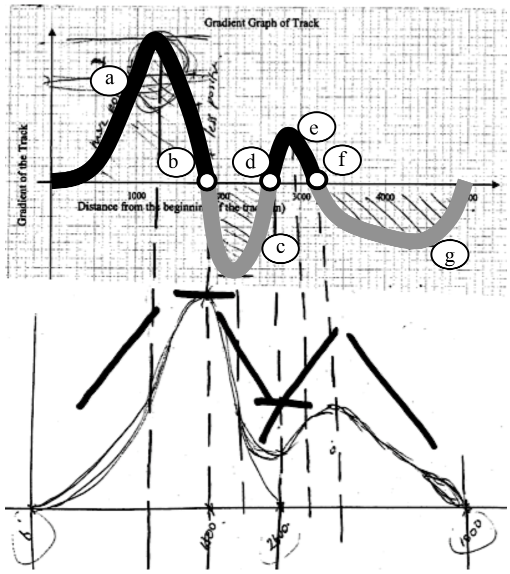

Figure 7. The “rough sketch” shown in thick black lines (enhanced from original).

Semiotic Bundle Stage 4

In the fourth stage of the semiotic bundle Noa describes a strategy for finding the peaks/valleys using a rough sketch (see thick black lines bottom right in Figure 7).

> [Noa, 250] Yeah, so this is a positive gradient [points to a], so it’s basically up [draws / on the graph], this is zero [points to b], it's across [draws – on the graph], this one is negative [points to c] it must be down in that whole portion [draws \ on the graph]. This one is zero [points to d] it has to be flat [draws – on the graph]. This one is positive [points to e] it has to be up [draws / on the graph] and this bit from here in this section is negative [points to g], it has to be down [draws \ on the graph]. So then you've got a bit of a sketch already of where the hills and valleys are.

Noa has used a graphical method to show where the “peaks and valleys” are on the graph of the antiderivative track. Although she hasn’t formalised it as a general verbal rule she has a local diagram showing where the maxima and minima are on this particular graph. Thus she uses the semiotic resource of the “sketch” to show where the maxima and minima are, using it as a tool that neglects the gradual changes in the steepness of the gradient to highlight the radical changes in the slope’s sense.

In terms of mathematical properties, the sketch, with its thick, black straight lines demonstrates that they were able to examine the antiderivative not just point by point but in a more holistic manner, matching the positive and negative sections of the gradient graph with whole sections of the antiderivative that have positive and negative gradients. They were also able to check that the zeros between the gradient sections correctly gave a horizontal tangent on the track. Thus the stripped down appearance of this sketch (compared to the graph previously drawn) helped them focus on creating a way to classify maxima and minima on the antiderivative track. It got rid of inessential features for this particular task, such as the accuracy in terms of height and subtle changes in slope. In this way they were able to formalise one of their strategies and come to reliable conclusions.

Discussion

In this paper we have described the changing nature of the semiotic bundle employed in the graphical construction of an antiderivative function and how multimodal interactions gave rise to mathematical knowledge production. This demonstrates that as the thinking of the two teachers about the constructs progressed it was distributed across the resources in the semiotic bundle. It would appear that they bring to the fore in their thinking the semiotic resources they perceive can play a significant role in building their understanding. This process is probably not clearly predicated on conscious reasoning, but a possible reason for moving across the resources could be to reduce cognitive load, which Sweller (1994) and others define as the demand imposed on an individual's cognition while facing a cognitive challenge. Graphical construction of antiderivative has proven to be such a challenge. One factor that increases intrinsic cognitive load is the number of elements that have to be kept in memory, and the interactions between them. In the case of graphical antiderivative the number of elements is high, including the local and global properties of the gradient graph and the implications of each for the antiderivative. Changing semiotic resources enables one to focus on fewer elements, such as the sign of the gradient or its magnitude, and thus reduce the load. In addition, demonstrating the versatility to change semiotic resources not only reduces the cognitive load but it also has benefits in terms of the affordances of the resource. In the case discussed here it seems clear that the gestures have dynamic affordance compared with the static graphical representations, more easily enabling a global perspective. This also highlights what we see as the difference between a passive representation in a given representation system, versus an active semiotic resource. The latter is goal-oriented and more related to learning through thinking about the task in hand. In addition, the unity of gesture and speech, as different sides of the same mental process with language and gesture comprising a single cognitive system (Arzarello & Robutti, 2008), gives rise to the affordance of strong communication of perceived constructs, in turn cementing them in the mind of the communicator.

Another advantage with regard to learning seen in this research was the link between the use of a problem in context designed to engage the participants in the mathematising part of the modelling process (Lesh & Doerr, 2003), and the embodied cognition that this gave rise to. The principle of the embodied mind, that mathematics is grounded in the human body is increasingly espoused (Lakoff & Nùñez, 2000; Arzarello & Robutti, 2008). Agreeing with this view, Tall (2008) includes embodied thinking in his model of the development of mathematical thinking, as a crucial world that can promote and encourage development of mathematical ideas. In this study the track context may have helped the teachers make considerable use of embodied notions, such as 'uphill' and 'downhill', 'steeper' and 'less steep' that seemed to assist with formalising these into mathematical constructs. At more than one point in the problem solving process the teachers found a novel way to decrease their cognitive load, and increase the affordances of their environment. They did this by the construction of a new semiotic system that we have called a virtual space. This is simultaneously a mathematical and a cognitive space linked to, and constrained by, the gesture space, and part of Arzarello's (2008) APC space. The virtual space provided affordances of communication, dynamic construction, an outlet for embodied gestures, and an environment to build and test conjectures. Construction and use of this virtual space required considerable versatility of mathematical thinking (Thomas, 2008). Firstly the teachers interacted with the gradient graph, employing it as a semiotic resource to discern properties of the antiderivative. Secondly the properties they discovered were translated into linked gestures in the gesture semiotic system, and these were then converted to a cognitive representation of the antiderivative in a new semiotic system, the virtual space. Finally this virtual representation was converted to a graphical representation of antiderivative. In our view the research presented here supports the view that mathematical thinking is multimodal and that as teachers we do well to take this into account in our teaching practices.

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