Modelling the Height of the Antiderivative

Caroline Yoon
The University of Auckland, NZ
<c.yoon@auckland.ac.nz>

Tommy Dreyfus
Tel Aviv University, Israel
<tommyd@post.tau.ac.il>

Michael O. J. Thomas
The University of Auckland, NZ
<moj.thomas@auckland.ac.nz>

Two complementary processes involved in mathematical modelling are (1) mathematising a realistic situation, and (2) applying a mathematical technique to a given realistic situation. We present and analyse work from four undergraduate students and two secondary school teachers who engaged in both processes during a mathematical modelling task that required them to find the antiderivative of a function presented graphically. When determining the height of the antiderivative, they mathematised the situation to develop an elementary mathematical method, and attempted to apply some ideas about definite integration that they had previously learned in class. In the end, however, the participants favoured their more elementary mathematised knowledge over the sophisticated knowledge they tried to apply.

Background

The words "Mathematical modelling and applications" have often been used to describe a set of instructional activities that involve the traversal between the "real world", and the "mathematical world". In general, the term "mathematical modelling" refers to the process of starting with a problem in the real world, and developing a mathematical model to solve the problem. In contrast, the term "applications" tends to denote the act of taking a known mathematical technique or idea, and using it in a problem that is set in the real world (see Figure 1).

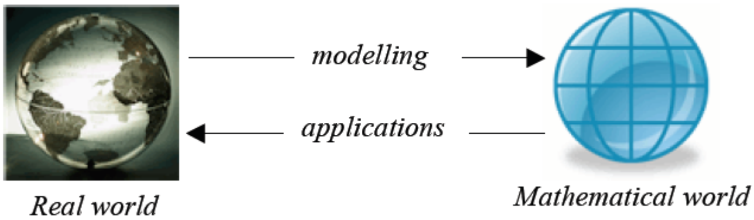

Figure 1. The difference between modelling and applications, adapted from Lesh & Doerr (p.4, 2003)

This characterisation is somewhat crude, however, as mathematical modelling actually involves a number of processes such as mathematising, interpreting, communicating, and even the process of applying (Maaß, 2006; Lesh & Doerr, 2003; Blum & Niss, 1991). These processes are often depicted in stages around the modelling diagram (see Figure 2), which students cycle around as they refine and develop their mathematical model. Researchers have noted that when students engage in mathematical modelling, they often go through multiple modelling cycles (Lesh & Doerr, 2003), thereby traversing between the real and model worlds multiple times. Thus, the mathematical modelling process is seen as a complex coordination of thinking processes that the learner engages in while travelling back and forth between real and mathematical worlds.

In contrast, problems that are classed as "applications" tend to be seen as involving a less advanced type of thinking. Applications-type activities typically refer to traditional word problems or exercises at the back of chapters in standard textbooks, where students are presented with a real-world context and are required to

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ascertain which mathematical tool they should apply. One key feature of this type of problem is that the student already knows the mathematical tool that the problem calls for, and only needs to make minor adjustments to it so it can be successfully applied. Consequently, researchers often consider the main thinking process involved in these kinds of problems as "undressing" the real-world context to decipher what the problem is asking for (Blum & Niss, 1991; Blum, 2002).

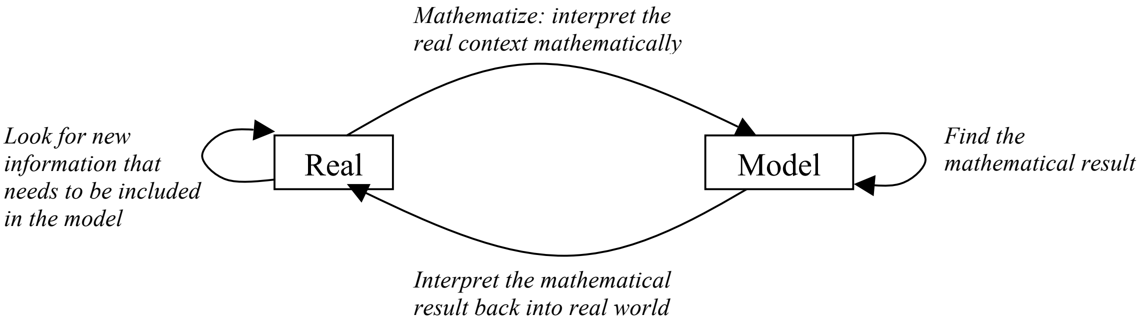

Figure 2. Elements of the modelling process.

In this paper, we consider two mathematical processes that are typically associated with "modelling and applications": (1) the process of *mathematising* a realistic situation, and (2) the process of *applying* a mathematical technique to a given realistic situation. We present data from students who engaged in both processes while working on a mathematical modelling activity that was set within the context of tramping, and which asked students to find the graphical antiderivative of a function presented graphically. As all of the participants had previously learned about differentiation and integration, this mathematical modelling activity gave them an opportunity to apply what they knew to solve the problem. We describe the students' attempts to mathematise the real world context, and compare their mathematisations with their attempts to apply their knowledge of integration to solve the problem.

The Mathematical Modelling Activity

The mathematical modelling activity was developed according to six principles for designing a particular class of activities called *Model-Eliciting Activities* (for a detailed explanation of these principles, see Lesh et al., 2000). In accordance with the *reality principle*, the problem was set in a context that could be personally meaningful to students outside the mathematical world. Students began by reading a newspaper article that discussed the inadequacy of difficulty ratings for tramping tracks around New Zealand, pointing out that these ratings did not give enough information about the steepness of tramping tracks. They then worked on some warm up activities in which they were given a distance-height graph of a tramping track, and asked to find its gradient graph (i.e., derivative). They were given a formula for calculating the gradient of tangent lines to assist them with the warm up activities.

After completing the warm up activities, participants were given the gradient graph of a tramping track shown in Figure 3, and were asked to develop a method for finding the distance-height graph of the original track. By asking them to develop a method instead of merely providing their solution, the activity fulfilled the *model construction principle*. In mathematical terms, this amounted to creating a method for finding an antiderivative of the given gradient graph, and the embedding of this mathematical task in the tramping context satisfied the *simple prototype principle*.

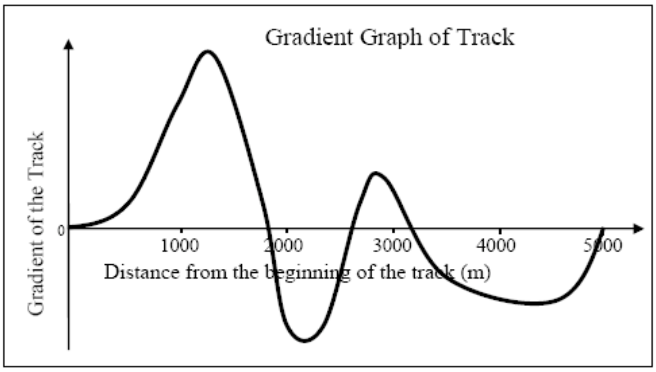

Figure 3. The gradient graph given in the modelling task.

The participants were asked to write their method for finding the graph of the original track in the form of a letter to clients who wished to determine whether the track was suitable for their purposes, thereby fulfilling the *model documentation principle*. In accordance with *model generalisation principle*, the activity asked the participants to design their method so that it works for any gradient graph, not just the one given. Finally, students were instructed to use their method to draw the distance-height graph, which gave them a chance to test and revise their method, thereby satisfying the *self-assessment principle*.

The Participants

There were six participants in this study, four of whom were enrolled in a first year undergraduate mathematics course at a large New Zealand university at the time. During the two weeks prior to working on this task, the four undergraduate students had completed a unit on integration techniques, and had earlier been introduced to differentiation techniques. The other two participants were secondary school teachers who had taught up to year 12 mathematics, but had not taught year 13 calculus. The participants were videotaped and audiotaped as they worked in pairs on the modelling activity over the course of an hour.

In this paper, we consider the work of Cam and Sid (two undergraduate students), in some detail. We also consider to a lesser degree, the work of Amy and Jay (two undergraduate students), and Ava and Noa (the two teachers).

Results and Analysis

While working on the modelling activity, Cam and Sid engaged in both processes of mathematising and applying when they addressed the question of the height of the antiderivative graph. We present excerpts from their activity in three stages, which represent three distinct attempts to draw the antiderivative (i.e., the track). We analyse their attempts to mathematise the tramping context to find the height of the track, as well as their attempts to apply their previously learned knowledge of integration to the tramping context.

First Drawing of the Antiderivative

Cam and Sid began by correctly determining that positive portions of the graph (marked as A and E in Figure 4) correspond to uphill portions of the track, whereas negative portions of the graph (C and G in Figure 4) correspond to downhill portions of the track. They also correctly ascertained that the x-axis intercepts on the graph indicate summits (B and F in Figure 4) and a valley (D in Figure 4) on the track.

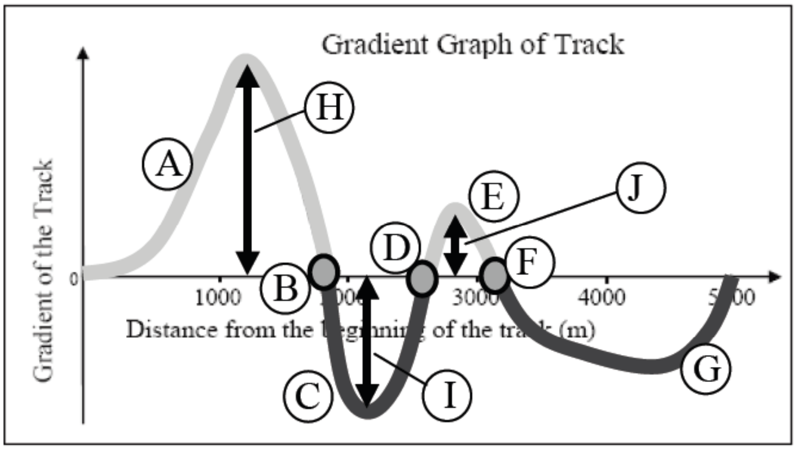

Figure 4. The sections (A-J) of the gradient graph to which Cam and Sid refer.

When they drew these features in their first distance-height graph of the track (shown in Figure 5), they assumed that the bottom of the valley is at sea level. They then considered the height of the track for the first time when they tried to determine the height of the second summit in relation to the first summit.

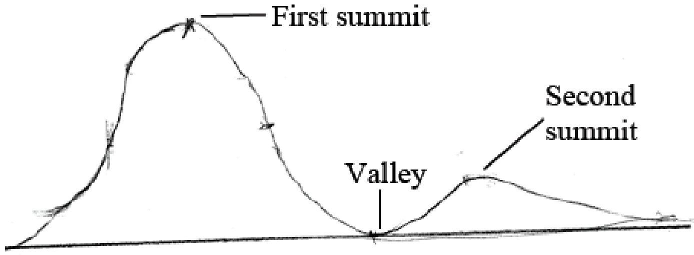

Figure 5. Cam and Sid's first drawing of the distance-height graph of the track (with labels added to indicate the first and second summits and valley).

Cam initially suggested that the second summit is one third of the height of the first summit, since the height of the second maximum on the gradient graph (indicated as J in Figure 4) is one third the height of the first maximum (indicated as I in Figure 4). Thus, in his first attempt to mathematise the height of the track, he inferred a proportional relationship between the heights of the local maxima on the gradient graph, and the heights of the summits of the tramping track:

$$\frac{\text{maximum}_2 \text{ (of gradient graph)}}{\text{maximum}_1 \text{ (of gradient graph)}} = \frac{\text{height of summit}_2 \text{ (of the track)}}{\text{height of summit}_1 \text{ (of the track)}}$$

However, he soon realised that this mathematisation was incorrect, and corrected himself, saying "Oh no it just gets a third as steep, that's got nothing to do with distance (points to J in Figure 4), that's not as steep, so it's just flatter" (note that in this excerpt, he uses the word "distance" to refer to the *height* of track). He explained that the height of the gradient graph indicates the steepness of the track, not the track's height. He then wondered what features of the graph would help him find the height of the track, and brought up an idea he remembered from class. Note that in the following excerpt, he again uses the word "distance" when referring to the *height* of the track:

Cam	What does the area under the gradient graph mean? Doesn't it mean something as well?
Sid	No because well you can't have negative areas so…
Cam	No but the absolute value of an area, isn't that distance? (Directs question to interviewer) Are we allowed to use our textbook?

Cam opened his textbook to the section describing "area under a curve", but dismissed it after finding it written in the context of speed. He decided that he could not apply his knowledge about the area under a curve to determine the height of the track at the second summit, because his textbook-based knowledge was situated in the context of speed, which he could not relate to the tramping context of the problem. Thus, Cam's initial attempts to mathematise the relationship between the gradient graph and the height of the antiderivative, and to apply his knowledge about integration were unsuccessful, and the question of how to find the height of the antiderivative remained.

Second Drawing of the Antiderivative

Later, Sid drew a "good copy" of the graph they drew in Figure 5, and again drew the bottom of the valley at sea level (see Figure 6).

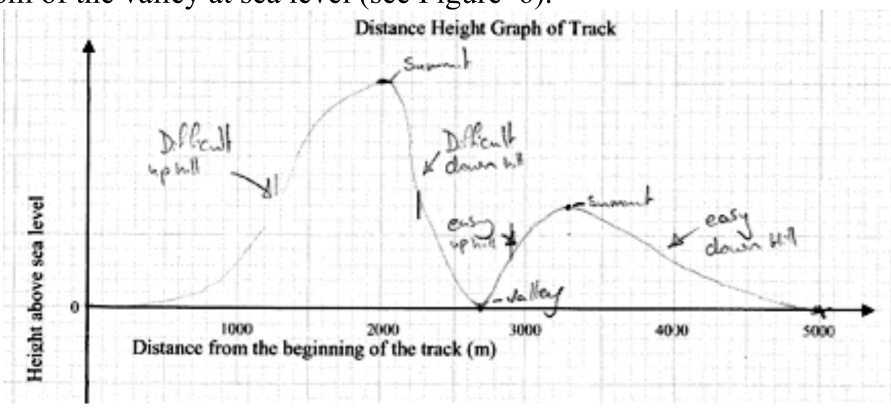

Figure 6. Cam and Sid's second drawing of the distance-height graph of the track.

While redrawing the graph, Cam and Sid revisited the question of the second summit's height in relation to the first. This time, Cam suggested that the second summit is not as high as the first summit because although sections E and A (in Figure 4) are "fairly similar shapes", E occurs over a shorter horizontal length than A. Thus, he correctly recognised that the horizontal distance is an important factor in determining the height of the antiderivative. He suggested that the summit that is reached by travelling at a shallow gradient over a short horizontal distance is not as high as the summit reached by travelling at a steep gradient over a long horizontal distance. Thus, Cam incorporated the horizontal distance into his second mathematisation of the relationship between the gradient graph and the height of the antiderivative. However, this mathematisation was not formalised until the third drawing of the graph.

Third Drawing of the Graph

After writing down their method in a letter, Cam observed that the steepest downhill portion of the track should only be "50% as steep" as the steepest uphill portion of the track, since the (absolute value of the) height of the first minimum on the gradient graph (I in Figure 4) is half that of the first maximum (H in Figure 4).

Thus, he inferred a proportional relationship between the heights of the first maximum and minimum on the gradient graph, and the steepness of the corresponding parts of the track.

Cam then redrew the track (see Figure 7) to reflect the differences in steepness, and in doing so also corrected the height of the bottom of the valley in the track with the comment: “I think what misled us was that this (points to the first downhill portion of the track in Figure 7) was over a shorter distance, so maybe it (points to the valley in Figure 7) doesn’t go all the way down to the ground.”

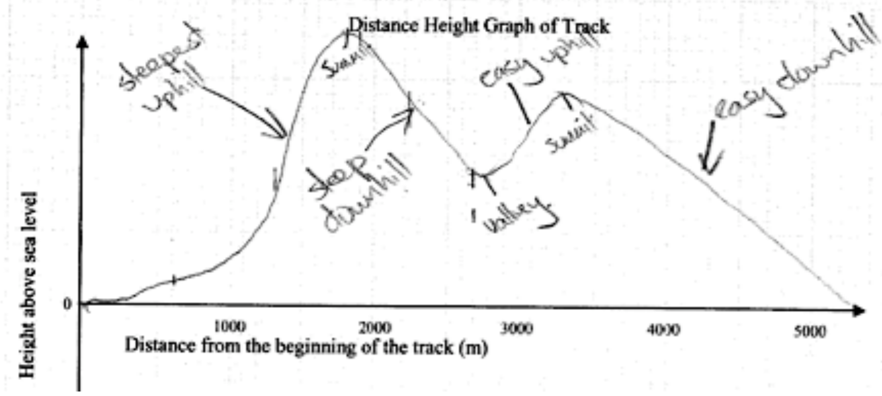

Figure 7. Cam and Sid’s third drawing of the distance-height graph of the track.

Next, Cam explained to Sid that the horizontal distance in the gradient graph is as important as the amplitude when determining the height of the track:

1	Cam	I think that the distance covered (referring to the horizontal distance) on the gradient graph does indicate the height climbed or descended in the…
2	Sid	Yup. I’d be more that the amplitude for this section here (points to C in Figure 4) is not as big as the amplitude here (points to A in Figure 4), so therefore it doesn’t reach ground level again.
3	Cam	Yeah, do you think it’s maybe a combination of the two, ‘cos if that was a lesser amplitude (points to C in Figure 4) but over the whole graph (points along the x-axis in Figure 4) then it would be going downhill the whole way?
4	Sid	Yeah but then we would see that as it goes the whole way, but just for that part…
5	Cam	Yeah, but for any graph. Just general graphs. The amplitude, it’s kind of that the amplitude times how far it’s gone is going to give you an indication of how high you’re going to go, how far you’re going to drop?”
6	Sid	Yeah.

Cam summarised this insight in a postscript to their letter: “you should take into account both the amplitude and horizontal distance as an indicator of the change in elevation for each slope”. Although Cam did not talk in terms of area under the curve, he commented in line 5 of the excerpt above that the change in elevation for the track is determined by “the amplitude times how far it’s gone”, or in other words:

$$\text{Change in elevation} = \text{amplitude of gradient graph} \times \text{horizontal distance}$$

This third mathematisation is essentially a crude approximation of the area under the curve using one Riemann rectangle for each area (see Figure 8). Thus, Cam initially dismissed the application of the “area under the curve” idea he had previously learned because it was embedded in a speed context in the textbook. However, he ended up developing an elementary understanding of the very same idea in his third mathematisation of the relationship between the gradient graph and the height of the antiderivative.

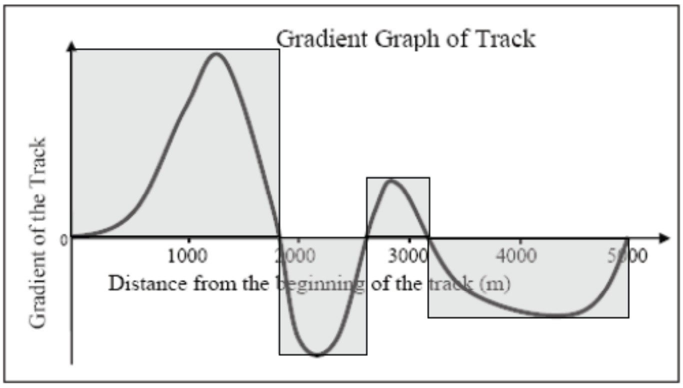

Figure 8. A graphical representation of Cam's insight into the relationship between amplitude of the gradient graph and the horizontal distance.

Similarities with Amy and Jay, Ava and Noa

Cam and Sid were not the only ones who tried to apply the idea of integration to the problem, but rejected it in favour of more primitive mathematisations. Both Amy and Jay, and Ava and Noa attempted to apply integration ideas and techniques, but in the end decided they could not be applied to the problem.

Amy and Jay started by remarking it was a pity they didn't have the algebraic equation of the given gradient function, since then they could easily solve the problem by integrating the equation. They considered using definite integration to find the area under the curve, and mentioned the rectangle method, upper and lower rectangles, and the trapezium method. However, they seemed unsure whether the application of these integration techniques would give them the height of the graph that they wanted. Interestingly, this uncertainty contradicts with their opening remark that they could have solved the problem by integration had they had the equation of the function, and suggests their understanding of integration was not distributed across algebraic and graphical representations. In the end, they used the symmetry of the gradient graph to argue about the height of the track, and declared that there was "no other way" to find the height, thus rejecting the rectangle method that they had earlier discussed.

Ava and Noa also considered what the area under the curve would give them, and whether they could apply integration to the problem. However, when trying to recall what they knew about integration, they drew graphs of speed, distance, and time. In a way that was similar to Cam, their knowledge of integration seemed to be rooted in the context of speed, and they were unable to apply it to the tramping context, and ultimately decided that the gradient graph did not show the height of the track.

Discussion

The students' failed attempts to apply their knowledge of integration to the tramping problem reveal two notable features of their previously learned knowledge. First, in the case of Cam and Sid, and Ava and Noa, their knowledge of integration was situated in the context of speed. They found it difficult to relate this context to the tramping context of the problem, and eventually abandoned their attempts to apply their knowledge of integration to the problem. This suggests that the teaching community's predominant use of the context of speed (Gravemeijer & Doorman, 1999) to illustrate integration concepts may be limiting students' ability to apply their knowledge of integration to other contexts.

A second interesting feature of the participants' knowledge of integration is their lack of representational versatility (Thomas, 2008). This was particularly telling in the case of Amy and Jay, who initially claimed they could solve the problem by integrating the algebraic equation of the gradient graph, but were later unsure whether numerical integration methods such as the rectangle and trapezium methods would indeed give them the height of the antiderivative. Thus, they were unable to link their algebraic representation of integration to numerical and graphical representations of the same concept.

Although the students' work on the modelling activity revealed serious limitations in their previously learned knowledge of integration, it also gave them an opportunity to build up their conceptual understandings through the iterative process of mathematising. Cam went through at least three cycles in the modelling diagram (Figure 2), and eventually mathematised the height of the track as the product of the amplitude of the gradient graph and the horizontal distance in the track. Cam's new knowledge was more elementary than the sophisticated integration ideas he had previously rejected. However, the process of mathematising enabled him to develop a better understanding of integration than he had previously demonstrated in his over-reliance on the textbook formulation.

A wealth of research suggests that although most students of calculus are reasonably proficient in performing various calculus techniques, they often lack a conceptual understanding of the core ideas (Thompson, 1994; Thomas & Hong, 1996; Eisenberg & Dreyfus, 1991). This paper suggests that students can benefit from mathematical modelling activities that encourage them to develop, express, test, and revise their own conceptual understandings of calculus concepts, particularly by engaging in the process of mathematising.

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