Aspects of Teachers' Knowledge for Helping Students Learn About Ratio

Helen Chick
University of Melbourne
<h.chick@unimelb.edu.au>

Ratio (and associated topics such as fractions and proportion) is known to be an area of mathematics that students find difficult. Multiplicative thinking is necessary, and students benefit from a wide range of strategies and representations for interpreting ratio. This study examined aspects of teachers' pedagogical content knowledge for teaching ratio, and investigated their knowledge of a typical misconception together with the strategies that they would use for dealing with such a misconception. The nature of the numerical examples that they suggested might be useful in teaching was also examined. Most teachers were able to recognise the misconception, but not all were able to generate examples that might help students to deal with it. Teachers also appeared to have only a limited repertoire of strategies to assist students.

Research into children's learning has long revealed that the topic of ratio—together with the allied topics of fractions, proportions, and percentages—is one that students find difficult (e.g., Hart, 1982). This is no surprise to teachers, whose experiences often lead them to list the topic as one of the more problematic in the curriculum. This small study examines the extent to which teachers can recognise a typical misconception associated with ratio understanding, and what strategies they have for addressing it. This implies an examination of their understanding of children's conceptions, their knowledge of models and explanations for teaching, and their capacity to modify examples in pedagogically useful ways, all of which lie in the domain of pedagogical content knowledge (PCK) (Shulman, 1986; see also Chick, Baker, Pham, & Cheng, 2006; Chick, 2007). The study also adds to the growing literature examining PCK using questionnaires with open-ended items that involve pedagogical situations; see, for example, the study of Watson, Beswick, and Brown (2006) which investigated teachers' knowledge about the teaching of fractions.

Background

One of the classic results in the history of examining students' understanding about ratio came from a large study of UK high school students reported by Hart (1982). In the Mr Short and Mr Tall problem, Mr Short was presented as being 6 paperclips tall, and then, when measured instead by matchsticks, was 4 match-sticks tall. Mr Tall, in contrast, was 6 matchsticks tall, and students were asked to determine Mr Tall's height in paper-clips. Among the approximately two-thirds of students who could not answer this question correctly, many made the additive error of focussing on 6 – 4 difference in the two measurements for Mr Short, and then added this to Mr Tall's height of 6 to get his paper-clip height as 8.

Many studies since have examined students' understanding of ratio more closely, often focussing on the challenge of developing the necessary multiplicative thinking, and on the importance of identifying the component parts and the associated whole. Norton (2004), studying 12-13-year-old students, used the context of gears in Lego construction kits to help build multiplicative thinking. Some incidental understanding of equivalence was developed, such as recognising that, for gears in the ratio of 13:24, turning the larger gear

L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia.* Fremantle: MERGA.

once will result in the smaller one turning 1.846 or approximately two times. Yeh and Nason (2008), working with adults with low numeracy skills, encountered a learner who struggled with even elementary equivalence such as between 1:1 and 2:2. In fact, it might be conjectured in this case that the learner's belief in non-equivalence actually led to him failing to see the "sameness" in the shades of paint used as physical representations of the ratios. The learners in the Yeh and Nason study also struggled to identify the whole. They thought that the number of "ingredients" comprised the whole, as opposed to the total number of parts present among all the ingredients, so that, for example, with four paint colours in the ratio 1:2:2:2 there was a belief that there were four parts (the number of colours) rather than nine (the number of actual parts).

The studies of Mitchelmore, White, and McMaster (2007) and Steinthorsdottir and Sriraman (2009) involved extended teaching sequences that were intended to focus on multiplicative relationships in general and ratio in particular. In the first study, with high school students, there was a focus on abstracting the idea of ratio from a series of contextualised situations. The researchers found that average ability students still had difficulty with understanding the connection between fractional and ratio representations, but noticed the usefulness of a visual depiction of ratio using a partitioned bar. The latter study, with older primary school students, showed how students' knowledge can develop with appropriate cognitively guided instruction, and identified different levels of understanding, including one associated with the difficulties of dealing with non-integer multipliers.

In addition to visual representations, another strategy that has been highlighted for ratio work is the use of ratio tables (Middleton & van den Heuvel-Panhuizen, 1994; also Brinker, 1998; Dole, 2008). A ratio table is an organising device that allows students to start with a known ratio and develop equivalent ratios, by using simple multiplication strategies, such as doubling and multiplying by 10, and then allows addition to combine previous results to obtain more complex ratios (thus, for example, a recipe involving a ratio like 2:3 might be "scaled up" to serve 12 times the amount by scaling the original by 10 (20:30) and also by 2 (4:6), and adding the results to get a ratio 24:36. This facilitates mental strategies, although it may be less effective at highlighting the direct multiplicative scaling by a factor of 12. In contrast, a teacher in one of the author's earlier studies (Chick & Harris, 2007; Chick, 2009) was very explicit in the way that she highlighted multiplicative relationships—using *only* multiplication, rather than a hybrid with addition—within an organising table like the ratio table. (See also further comments in Chick (2009) concerning the kinds of numbers that should be used as multipliers to ensure that students understand the full generality of equivalence.)

In fact, the question of how choice of numbers affects what examples might be able to exemplify is highlighted in the work of Watson and Mason (2005, 2006). Drawing on the work of Ference Marton, they highlight that general principles can be made apparent through the judicious variation of parameters or dimensions of the situation. This notion of *dimensions of variation* was explored, implicitly, in the study of two teachers' choices of examples for introductory work in ratio at the upper primary school level (Chick & Harris, 2007; Chick, 2009). It seemed evident that the capacity of the lessons to illustrate important ratio principles was influenced heavily by the numerical values and contexts of the examples used. Skemp (1971, pp. 29-30) also highlights the importance of signal and noise in examples, implying that teachers' examples must allow the key principle to be identified. Thus it seems that specific choices among the dimensions of variation will impact on an example's pedagogical effectiveness (see also Chick, 2007).

Method

The present research examined aspects of teachers' knowledge for teaching ratio. It was part of a larger study, involving a questionnaire and interview protocol intended to explore the PCK of secondary teachers who were teaching mathematics. There were 40 teachers from three schools involved in the study, comprising all those teaching at least one mathematics class in their school. The teachers' backgrounds ranged from being first year graduates and non-specialists through to teachers with many years of experience including teaching senior level mathematics. Each teacher completed a questionnaire in his/her own time, with the questionnaire focussing on a number of key topics from Years 7 to 9 of the mathematics curriculum. The fourth item in the questionnaire addressed ratio, and comprised the stem and questions shown in Figure 1.

The following question was given to Year 8 students:

> Some children are making pink paint by mixing together white and red.
> Shea uses 4 spoonfuls of red paint and 11 spoonfuls of white paint.
> Mi-Lin uses 6 spoonfuls of red paint and 13 spoonfuls of white paint.

(a) One student thinks these paint mixes will look the same shade of pink. Why might he think this?
(b) What assistance/explanations might you provide to such a student?
(c) What changes to the numbers in the question might you make in order to help students?

Figure 1. Questionnaire item about ratio given to teachers.

It should be noted that an early version of the questionnaire had question (c) worded more generally as "What examples might you use to help?". A follow-up interview was conducted with each teacher once the whole questionnaire had been completed. This interview, which followed a semi-structured protocol, gave teachers the opportunity to expand on and clarify their written responses. There were three standard questions, listed below, and occasionally teachers were asked additional questions depending on their particular responses.

- Do you think many students would think that these paint mixes give the same shade of pink?
- What difficulties do they [students] have with such problems?
- [With reference to (c)] Why did you pick those examples?

The audio-recorded interview responses were transcribed. This was not a verbatim transcription; instead responses were paraphrased to capture the main themes and specific examples. The questionnaire responses and transcribed interview responses were then incorporated into a spreadsheet. Content analysis (Bryman, 2004) was conducted with these data. Responses to question (a), together with the interview question about the prevalence of such a misconception, provided data to determine the extent to which the teachers were aware of the additive error misconception (and any other difficulties with ratio). Responses to (b) plus discussion arising in the interview provided evidence for the kinds of strategies and explanations that teachers might use to assist students. In this case the data were examined to identify emerging themes, which were then used to categorise the approaches. Finally, the whole data set was examined to collate the numerical examples that teachers proposed to use. These were analysed to determine whether or not the examples would address the additive error misconception or whether they were intended to assist with ratio understanding more generally. Some of the teachers' examples were selected as exemplifying cases (Bryman, 2004, p.51), in order to illustrate the impact of variation on what can be exemplified.

Although the data included information about teachers' experience and main teaching areas, there were a large number of categories within the small data set that made it impossible to conduct a satisfactory examination of whether teacher knowledge depends on experience or area of expertise. In fact, an informal overview of the data set suggests that there were inexperienced teachers teaching mathematics 'out of area' who gave 'good' examples and explanations, and experienced specialist teachers whose responses had shortcomings, and that opposite cases were also present.

Results and Discussion

Identifying Possible Misconceptions

Of the 40 teachers, over 80% (33 teachers) recognised that a possible reason for the student to believe that the two different paint mixes were the same shade of pink was because of an additive relationship between the two ratios. Most of them saw that the second ratio (Mi-Lin's) involved numbers that were two more than each of those in the first, whereas six teachers identified that both ratios had an internal difference of seven between the two parts. Of the remaining seven teachers—that is, those who had not identified the additive error—four thought that students might say "Red and white always make pink" because the students did not understand that the proportions matter (this was echoed by three of the teachers who also recognised the additive error), one pointed out that in both cases white is roughly twice the red, and two did not propose a possible reason for the misconception. Just under half of the teachers thought that the additive error would be a common misconception (but it must be noted that this question was not asked consistently and, of course, not all teachers had noticed the phenomenon).

Four teachers commented that literacy issues might affect students' capacity to engage with the problem, but all but one of these could also identify the additive error misconception as a possible explanation. This fourth teacher was also one of the three teachers who did not identify the additive error among the four teachers who claimed that the context might be difficult for some students. These four teachers suggested that students' lack of experience with paint mixing might imply that they would not comprehend the way different shades of colour arise. There is a temptation to draw parallels between the teachers' claim that students cannot engage in the mathematics because of the context, and these instances of teachers themselves not engaging with mathematical aspects of the learning process because of a context.

Proposed Teaching Strategies

Table 1 shows the different strategies for assistance or explanations that teachers proposed in response to question (b), supplemented by those strategies added during the interviews. In some cases a particular strategy was applied only to the specific paint-mixing ratio problem mentioned in the item, in other cases the same strategy was intended to develop understanding about ratios more generally. This is indicated by the phrase "general or problem-specific". It also should be noted that here "use other ratio examples" *only* includes those instances where teachers made explicit mention of this strategy in response to question (b), and does not count the examples suggested in response to the more specific question (c).

Table 1
Teaching Strategies Proposed by the Teachers (n = 40)

Teaching strategy	Number (%)
Common denominator/equivalent fractions (general or problem-specific)	7 (18%)
Pictorial representation (general or problem-specific)	15 (38%)
"Tell" or "Explain" (general or problem-specific)	6 (15%)
Convert to decimals (general or problem-specific)	9 (23%)
Unitary (general or problem-specific)	2 (5%)
Identify approximate relationship (e.g., relate to half) (problem-specific)	4 (10%)
General – use analogy (e.g., cooking)	8 (20%)
General – do mixing with real materials	12 (30%)
General – use other ratio examples	7 (18%)
General – vague	5 (13%)

Most of the suggestions from the 15 teachers for some kind of pictorial representation were problematic. There were four circular area models (pie "graphs" or pizzas), for which both the production of the representations and comparisons are difficult. Four suggestions were to use discrete materials, which are limited in their capacity to develop equivalent wholes (so that, for example, to compare 2:5 and 3:7 it is necessary to find equivalent ratios for each that can then be compared more readily). Five teachers proposed undescribed "diagrams", drawings of tins, or use of coins. Finally, one teacher suggested a "colour picker", in which graphics programs mix colours based on numerical values for different colours, but the proposer did not realise that most implementations of these do not use ratios (e.g., the colour produced by, say, an RGB value of 1-1-1 is different from that produced by 20-20-20). Only two of the suggestions were for number lines, which are both easy to apportion and compare (one suggestion was from a teacher who had also proposed a pie diagram, hence the 16 proposals from only 15 teachers). No teacher proposed the use of ratio tables or similar. Among the suggestions to mix real materials were paint (2), cordial and water (4), and food colouring (1), with the remainder unspecified but possibly implying paint (5). Little if any mention was made of the practicalities of this, and whether or not close ratios could be distinguished in practice. The six "tell" or "explain" responses gave no details about what might be emphasised in such an explanation.

Variation in the Proposed Examples

There were eleven teachers who did not give any additional specific numerical ratios that they thought might be useful for students to consider. Although the vague wording of the early version of question (c) may have contributed to this, by not emphasising that the researcher was interested in changes to the numbers used in the examples, most of the follow up interviews clarified this intention and gave teachers a chance to respond. In at least three cases, the researcher probed specifically for an example that would meet certain requirements but the teachers talked around the issue rather than giving numerical values.

Seven of the teachers continued to use the values given in the original problem to illustrate points that they wanted to make. For example, one teacher wrote "I might reduce the numbers so they could see that 11 is almost 3 times 4 and that 13 is only just twice 6", when in fact the explanation is satisfactory for the task as specified and he did not actually reduce the numbers at all. Another teacher asserted that she preferred the question as stated because it allows her to ascertain which students really have understood the ratio concept.

The examples from twelve of the teachers seemed to be designed to highlight the role of factors in simplifying ratios. As an example of a non-specific low-level response one teacher wrote that she would work with "fives and tens". In most cases, however, specific examples were given, and, in fact, these seemed part of an intention to address general ratio principles. One teacher, for instance, gave 5:15 and 3:12 as two examples that she might consider; similarly another teacher proposed 4:12 and 6:18. These choices allow for easy simplification and then comparison, although all of these examples involve simple reduction to a unitary ratio (in contrast to something like 8:12, which is equivalent to 2:3).

In fact, almost half (19) of the teachers gave examples that appeared to be concerned with broader ratio issues like equivalence and comparison (and some of these are among those mentioned in the previous paragraph involving obvious factors). One teacher chose 1:5 and 2:4, both of which have the same number of parts in total. Another chose examples that would allow students to relate ratios to benchmarks, suggesting that examples such as 5:10, 11:20, and 7:12—representing 1:2, and, for the latter two, a little more than 1 to 2—might force students to focus on ways of making comparisons. Other teachers focussed on doubling and halving operations and the fact that these lead to equivalent ratios.

Finally, and significantly, 15 of the teachers suggested examples that addressed the additive error issue (including one teacher who probably constructed his inadvertently because he had not actually identified the misconception). It should be noted that the researcher did not always probe for a misconception-addressing example in the case of teachers who had identified the error but whose examples did not target it. This may explain why less than half of the teachers who identified the misconception produced examples that addressed it. On some occasions, however, probing occurred and teachers varied in their capacity to respond 'on the spot'.

It is illuminating to explore the consequences of the differing numerical choices among the 15 error-addressing examples. One teacher tried to work with the original problem, explaining that adding a 4:11 mix and a 2:2 mix would have different proportions to the original 4:11 mix, but did not attend to the complex issue of what happens to wholes in this process. Three of the teachers gave 1:2 and 2:3 as example pairs: these are simple ratios, with an increment of one from the first to the second (i.e., both red and white have increased by one spoonful), but only one of the teachers specified the argument needed to convince students that these ratios are, indeed, different, saying that in the first ratio white is twice red, whereas in the second it is not. A similar 'what is doubled' relationship argument was proposed by the teacher who suggested using 1:3 and 2:4 (again, with an increment of one), and by the teacher who suggested 2:1 and 4:3 (this time an increment of two). One teacher proposed 4:12 and 8:16 (with an increment of 4), with the choice of numbers readily allowing simplification to obtain the easy-to-compare unitary equivalent ratios 1:3 and 1:2. Another teacher began with 4:11 and argued its equivalence to 8:22, and then highlighted that this is a different ratio from going up by two twice, which would give 8:15. A particularly striking example of an error-addressing pair of ratios was given by another teacher who proposed 1:10 and 11:20 (an increment of 10), with the former very obviously containing very little red—one-tenth of the white, in fact—and the latter containing about half as much red as white.

The final example to be discussed was constructed during the interview and shows a range of aspects of teacher knowledge. The teacher began by proposing 2:3 and 3:4 as an error-addressing pair, but then realised that the 3:4 in particular would be awkward to represent using her preferred sectors-of-a-circle model for fractions and ratios. The 2:3, with 5 parts in total, was not a problem since 360 is divisible by 5. She then adjusted 3:4 by another increment of 1 to make it 4:5, which is still related to 2:3, but which is now made of 9 parts and is thus easy to represent using the circle model. Her discussion in the interview articulated the mathematical relationships she was trying to achieve.

Conclusions

This study found that the vast majority of mathematics teachers—at least in this sample—recognised the additive error misconception. There was wide variation, however, in the teachers' suggestions for strategies that might assist students. Very few of the suggestions proposed in part (b) specifically addressed the additive error, although by the end of the overall questionnaire and interview protocol about half of the teachers who had recognised the additive error could provide targeted examples for students. It is acknowledged that the lack of detail about what, exactly, would be included in an "explain" or "tell" response may have been an artefact of the questionnaire approach and the lack of further probing in the interview. Nevertheless it may reflect a difficulty that some teachers have in actually articulating the mathematical heart of a teaching issue. If this is the case, then there is cause for concern here. Similarly, there were problems involving the mathematical and pedagogical heart of most of the pictorial representations as well. Pie 'graphs' may represent the mathematics appropriately but construction is relatively complex mathematically and they are not easy to compare, which is a pedagogical issue. Discrete models have difficulty representing a whole, and have limitations for showing equivalence depending on the numbers of parts involved. Finally, it was striking that no teacher proposed a simple computational organiser like a ratio table.

The examination of the examples that teachers proposed as ways of modifying the problem for teaching highlights how what it is possible to exemplify may be dramatically affected by the choice of numerical values. Some of the examples that teachers proposed seemed much better for featuring a principle than others.

These results highlight the complexity of the knowledge required for effective teaching. There is a need for mathematical knowledge, such as, for ratio, understanding the relationship between parts and wholes, and the concept of equivalence. There is a need to know appropriate models that have sufficient epistemic fidelity (cf. Stacey, Helme, Archer, & Condon, 2001) to represent the concepts in mathematically appropriate and pedagogically productive ways. There is a need to understand the pedagogical effect of the dimensions of variation in families of examples. There are, of course, other requirements as well, but these are particularly evident in this study.

The results also highlight how mathematical considerations have important pedagogical consequences. This is frequently assumed as being both true and something that those involved in teaching mathematics take into account, but there is evidence in the results here that, at the least, teachers' conversations about their work do not always focus on this. Helping educators to attend to these issues seems to be an important role of pre- and in-service professional learning activities.

Acknowledgements

This research was supported by Australian Research Council grant LP0882176 (and also informed by work from DP0344229), with co-researchers Kaye Stacey and Vicki Steinle, and research assistants Beth Price and Eugene Gvozdenko. Beth conducted some of the interviews.

References

Brinker, L. (1998) Using recipes and ratio tables to build on students' understanding of fractions. *Teaching Children Mathematics, 5* (4), 218-224.

Bryman, A. (2004). *Social research methods* (2nd ed.). Oxford: Oxford University Press.

Chick, H. L. (2007). Teaching and learning by example. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, pp. 3-21). Sydney: MERGA.

Chick, H. L. (2009). Choice and use of examples as a window on mathematical knowledge for teaching. *For the Learning of Mathematics, 29* (3), 26-30.

Chick, H. L., Baker, M., Pham, T., & Cheng, H. (2006). Aspects of teachers' pedagogical content knowledge for decimals. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Prague: PME.

Chick, H. L. & Harris, K. (2007). Pedagogical content knowledge and the use of examples for teaching ratio. *Proceedings of the 2007 annual conference of the Australian Association for Research in Education* (Fremantle, WA, 25 - 28 November, 2007). Retrieved 24 May 2010 from http://www.aare.edu.au/07pap/chi07286.pdf.

Dole, S. (2008). Ratio tables to promote proportional reasoning in the primary classroom. *Australian Primary Mathematics Classroom*, *13*(2), 18-22.

Hart, K. M. (Ed.). (1982). *Children's understanding of mathematics: 11-16*. London: John Murray.

Middleton, J. A., & van den Heuvel-Panhuizen, M. (1995). The ratio table. *Mathematics Teaching in the Middle School, 1*(4), 282-288.

Mitchelmore, M., White, P., & McMaster, H. (2007). Teaching ratio and rates for abstraction. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, pp. 503-512). Sydney: MERGA.

Norton, S. J. (2004). Using Lego construction to develop ratio understanding. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, pp. 414-421). Sydney: MERGA.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Skemp, R. R. (1971). *The psychology of learning mathematics.* Harmondsworth, UK: Penguin.

Stacey, K., Helme, S., Archer, S., & Condon, C. (2001). The effect of epistemic fidelity and accessibility on teaching with physical materials: A comparison of two models for teaching decimal numeration. *Educational Studies in Mathematics, 47*(2), 199-221.

Steinthorsdottir, O. B., & Sriraman, B. (2009). Icelandic 5th-grade girls' developmental trajectories in proportional reasoning. *Mathematics Education Research Journal, 21*, 6-30.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples.* Mahwah, NJ: Lawrence Erlbaum Associates.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91-111.

Watson, J., Beswick, K., & Brown, N. (2006). Teachers' knowledge of their students as learners and how to intervene. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 551-558). Sydney: MERGA.

Yeh, A., & Nason, R. (2008). Mixing colours: An ICT tool based on a semiotic framework for mathematical meaning-making about ratio and fractions. In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, pp. 605-611). Sydney: MERGA.