Enactivism and Figural Apprehension in the Context of Pattern Generalisation

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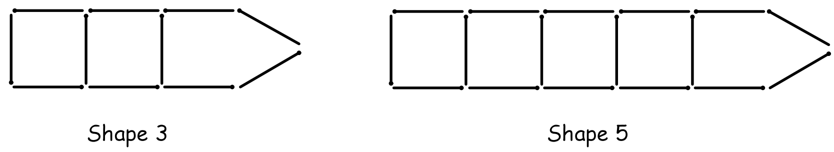
This paper seeks to establish a research framework for an investigation into the extent to which pupils are able to visualise figural cues in multiple ways within the context of pattern generalisation. Enactivism, along with the constructs of knowledge objectification and figural apprehension, are identified as forming an ideal theoretical framework for such a study. Although largely theoretically driven, this paper also presents results from an initial pilot study in order to contextualise the theoretical milieu.

Numerous mathematics educationalists and researchers have advocated a multiple representational view of pattern generalisation within the classroom - not only to explore the notion of *equivalence*, but to encourage pupils to critically engage with the underlying physical structure of the pictorial context as seen from alternative viewpoints.

By way of example, a typical pedagogical strategy is to suggest various equivalent algebraic expressions for the general term of a pictorial pattern. Pupils are then encouraged to arrive at plausible explanations for each expression by referring to the physical structure of the provided context. Figure 1 illustrates a typical example.



Look at the diagrams shown above. Shape 3 requires 11 matchsticks while shape 5 requires 17 matchsticks.

(a) Write down an algebraic formula that will determine how many matchsticks are needed to build the n^{th} shape.

(b) Terry says the formula is $2(n+1)+n$ while Jeff says the formula is $1+3(n-1)+4$. By referring to the diagram, explain how Terry and Jeff arrived at their respective formulae.

Figure 1. A typical question encouraging a multiple representational view of pattern generalisation.

While such a pedagogical approach may be useful for some pupils, for others it may well create additional complications. Not only is there an inherent ambiguity in the structure of the algebraic expressions – in the above example, $2(n+1)$ represents 2 multiples of $(n+1)$ matches while $3(n-1)$ represents $(n-1)$ multiples of 3 matches – but an additional cognitive burden is placed on pupils who write algebraic expressions in non-standard format (e.g., $n \times 3$ as opposed to $3n$).

This paper approaches multiple representations from a more fundamental level, and seeks to investigate the extent to which pupils are able to visualise figural cues in multiple ways within the context of pattern generalisation. A literature review reveals that very little empirical research has been done in this area. This paper forms part of a broader study which seeks ultimately to shed light on the embodied processes that either assist or hinder

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pupils' abilities to visualise figural cues in multiple ways within the context of pattern generalisation.

Visualisation

Visually mediated approaches to pattern generalisation tasks set within a pictorial context provide for an interesting interplay between two different modes of visual perception: sensory perception and cognitive perception (Rivera & Becker, 2008). These different modes resonate with Fischbein's (1993) theory of figural concepts, and the notion that all geometrical figures (or *figural objects*) possess, simultaneously, both conceptual and figural properties. Mariotti (as cited in Jones, 1998) stresses the dialectic relationship between figure and concept as an important interaction in the field of geometry, a relationship that can create tension from a student's perspective. It is suggested that a similar tension is likely to underlie visual strategies applied to pattern generalisation tasks set within a pictorial context.

In an attempt to elucidate this visual tension, primary importance will be placed on the notion of figural *apprehension* as espoused by the French psychologist Raymond Duval (1995). Although Duval's notion of figural apprehension was developed within a more classical geometry context, it can, with only slight modifications, be adapted readily to other contexts involving geometrical figures. Four different modes of apprehension of a figure are pertinent to this paper - perceptual, sequential, discursive and operative.

Perceptual apprehension refers to the initial apprehension of a figure - what we see in a perceived figure at first glance as determined by the unconscious integration of Gestalt laws of figural organisation. *Sequential apprehension* relates to the emergence of sub-figures or elementary figural units, which stem from either the construction of the perceived figure, or a description of its construction. *Discursive apprehension* is a process of perceptual recognition during which certain gestalt configurations gain prominence due to an association with discursive statements accompanying the geometric figure. *Operative apprehension* relates to the various ways by which a given figure can be modified while retaining the properties of the figure, for example by a reconfiguration of the whole-part relation through a recombination of elementary figural units.

Figure 2 shows possible outcomes for each of the four modes of figural apprehension based on the given visual stimulus. Perceptual apprehension subdivides the given figure into squares and triangles based on the Gestalt laws of closure and good form. This could potentially lead to a complicated algebraic expression for the general term, which would need to take into account overlapping matches: $T_n = 4n - (n-1) + 3(2n) + 6 - (2n+2)$. Sequential apprehension could arise from noticing that the construction of each subsequent term requires the insertion of a 7-match additive unit. This has the potential to yield the general expression $T_n = 7n + 5$. Discursive apprehension may be invoked by accompanying the visual stimulus with the wording "for 2 squares you need a total of 19 matches", thus foregrounding the structural unit of the square. This could potentially yield the general expression $T_n = 4n - (n-1) + 2(2n+2)$ where, after counting squares and adjusting for overlapping units, the remaining matches are seen as V-shapes around the perimeter. Finally, operative apprehension may allow the visual stimulus to be seen in terms of horizontal lines, vertical lines, V-shapes above and below, and a constant 2 matches at either end. Expressed algebraically, this yields $T_n = 2n + (n+1) + 4n + 4$.

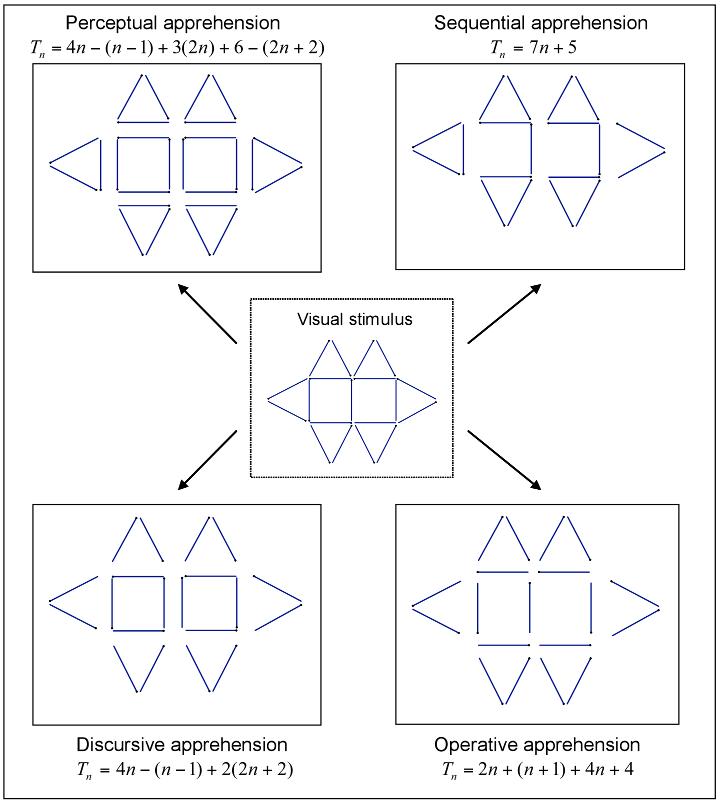


Figure 2. Possible outcomes for different modes of figural apprehension.

Duval (1998) makes the pertinent point that most diagrams contain a great variety of constituent gestalts and sub-configurations – far more than those initially identified through perceptual apprehension, or those made explicit through construction or accompanying discursive statements. Critically, it is this surplus that constitutes the *heuristic power* of a geometrical figure since specific sub-configurations may well trigger alternative solution paths.

Within the context of pattern generalisation, perceptual apprehension may on occasion be sufficient to generalise a given figural pattern. However, perceptual apprehension will not necessarily evoke gestalts, which are appropriate to the generalisation process. An inability to move beyond mere perceptual apprehension of a figure can lead to what Duval (1999) refers as *heuristic deficiency*. In order to actualise the heuristic potential of a diagram it is necessary not only to be aware of the scope of the diagram but also to be able to use it flexibly. Being able to see a diagram in multiple ways thus necessitates a move beyond perceptual apprehension. From a cognitive standpoint, this raises an important question: What are the embodied processes that either hinder or assist pupils in moving flexibly between different modes of apprehension?

Within the context of figural pattern generalisation, the processes of visualisation and generalisation are deeply interwoven. Pattern generalisation rests on an ability to *grasp* a commonality from a few elements of a sequence, an awareness that this commonality is applicable to *all* the terms of the sequence, and finally being able to use it to articulate a direct *expression* for the general term. There are two important aspects of this notion of generalisation: (a) a *phenomenological* element related to the grasping of the generality, and (b) a *semiotic* element related to the sign-mediated articulation of what is noticed in the phenomenological realm (Radford, 2006). The research methodology and underpinning theoretical framework need to be sensitive to both these aspects of figural pattern generalisation. Enactivism along with the theoretical construct of *knowledge objectification* (Radford, 2008) have been identified as ideally meeting this critical requirement.

Enactivism

Enactivism is a theory of cognition that draws on ideas from ecology, complexity theory, phenomenology, neural biology, and post-Darwinian evolutionary thought. The basic tenet of enactivism is that there is no division between mind and body, and thus no separation between cognition and any other kind of activity. Enactivist theory brings together action, knowledge and identity so that there is a conflation of doing, knowing, and being (Davis, Sumara, & Kieren, 1996). Within an enactivist framework, cognition is not seen as a representation of an external world, but rather as "an ongoing bringing forth of a world through the process of living itself" (Maturana & Varela, 1998, p. 11). Thus, cognition is viewed as an embodied and co-emergent interactive process where the emphasis is on *knowing* as opposed to *knowledge*.

For the enactivist, the act of perceiving something is not a process of recovering properties of an external object rather, "perception consists of perceptually guided action" (Varela, as quoted in Lozano, 2005, p. 26). Thus, we perceive things in a certain way because of the manner in which we relate to them through our actions (Lozano, 2005). This idea is succinctly stated in Maturana and Varela's 1998 aphorism: "*All doing is knowing, and all knowing is doing*" (p. 26). Thus, knowledge depends on "being in a world that is inseparable from our bodies, our language, and our social history – in short, from our *embodiment*" (Varela et al., 1991, p. 149). For Varela et al. (1991), who build on Merleau-Ponty's phenomenology, it is critical that we see our bodies as both physical (biological) structures as well as lived, phenomenological structures. Thus, *embodiment* has an important double sense: "it encompasses both the body as a lived, experiential structure and the body as the context or milieu of cognitive mechanisms" (Varela et al., 1991, p. xvi).

As Davis (1995) comments, language and action are not merely outward manifestations of internal workings, but rather they are "visible aspects of … embodied (enacted) understandings" (p. 4). For Davis et al. (1996), enactivism prompts us not only to consider the formal mathematical ideas that emerge from action, but to give close scrutiny to those preceding actions – "the unformulated exploration, the undirected movement, the unstructured interaction, wherein the body is wholly engaged in mathematical play" (p. 156), As Núñez, Edwards and Matos (1999) argued, the nature of situated cognition cannot be fully understood by attending only to the social and contextual factors. Learning and thinking are also situated "within biological and experiential contexts, contexts which have shaped, in a non-arbitrary way, our characteristic ways of making sense of the world" (Núñez, Edwards, & Matos, 1999, p. 46).

From an enactivist stance the perception of a given figural cue is not merely a visual process. Rather, perception needs to be considered as a fully *embodied* process. It is thus vital that the research methodology, both in terms of data capture and data analysis, takes appropriate cognizance of this embodied notion of perception. It is this issue that the following section seeks to address.

Perception and Knowledge Objectification

Perception is a complex cognitive activity related to the *manner* of our acquaintance with the objects of perception, in other words the *activity* that mediates our experience with objects (Radford, Bardini, & Sabena, 2007). An interrogation of the *embodied* processes of perception thus needs to focus on the phenomenological realm of students' experience in order to emphasize the subjective dimension of knowing (Radford, 2006). Radford (2008) refers to the process of making the objects of knowledge apparent as *objectification*, a multi-systemic, semiotic-mediated activity during which the perceptual act of noticing progressively unfolds. The objects, tools, linguistic devices and signs used by individuals in social meaning-making processes to achieve a stable form of awareness, he refers to as *semiotic means of objectification*. Such semiotic means of objectification could include: Words and linguistic devices; metaphor and metonymy; gestures; rhythm in speech and gesture; graphics and the use of artefacts.

Such a multi-semiotic view of knowledge objectification takes cognisance of the *principle of non-redundancy*, the notion that different semiotic systems allow for different forms of expressivity (Radford, 2006). In addition, it displays an enactivist sensitivity for the process of objectification in which the interplay of a variety of semiotic means/systems is seen to have a fundamental role in knowledge formation and in which cognitive activity is seen as being "… embodied in the corporality of actions" (Radford et al., 2007, p. 508). The theoretical construct of knowledge objectification is thus ideally suited to an enactivist theoretical framework in which there is a purposeful conflation of *doing*, *knowing*, and *being*.

Of particular interest to the research project are those pivotal moments when pupils move between different modes of apprehending a given figural pattern. By means of careful analysis of the data stemming from various semiotic means, this research seeks ultimately to characterise and provide rich descriptions of these pivotal moments.

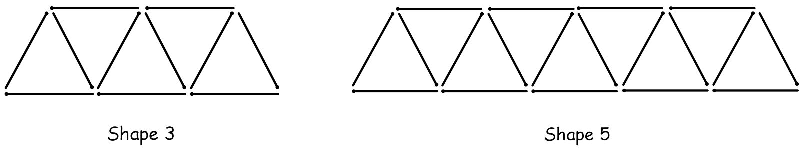
Methodology

This study is oriented within the conceptual framework of qualitative research, and is anchored within an interpretive paradigm. The research makes use of an instrumental case study approach, the research participants being a mixed gender, high ability Grade 9 class of approximately 25 learners from an independent school in Grahamstown, South Africa. This non-probability purposive sampling is supported by experience from previous research (Samson, 2007), which suggests that high ability learners are more likely to constitute "information-rich cases" (Patton, 1990, p. 169) given the data collection protocol and the nature of the patterning tasks under consideration.

From an enactivist methodological stance, this study is characterised by the use of multiple perspectives and the continuous refinement of methods and data analysis protocols over both time and form (e.g., audio-visual data examined repeatedly in different forms (video and transcript) and in conjunction with additional data retrieved from field-notes and participants' worksheets). The use of multiple data sources and approaches to data handling is in turn a form of triangulation.

Results

In an initial pilot study, individual participants were provided with two non-consecutive terms of a figural pattern and were asked to provide, in the space of one hour, different visually mediated expressions for the n^{th} term of the pattern. The pilot study suggests that some pupils have a surprising facility to visualise figural cues in multiple ways.



Visually mediated expressions for the nth term suggested and justified by a single research participant:

$3n+n-1$	$3(n-1)+4+n-2$	$2n+n+n-1$
$4n-1$	$n+n+n+[n-1]$	$2(2[n-1])+3$
$4(n-1)+3$	$n+3(n-1)+2$	$3n+3(n-1)-[2n-2]$

Figure 3. Figural pattern from pilot project.

There is a remarkable diversity in the algebraically equivalent expressions shown in Figure 3, each of which was suggested and fully justified by a single research participant (Grant), and arrived at through different modes of visual apprehension. What was particularly revealing was a meta-analysis of the semiotic means of objectification employed by respective research participants. The following vignette of one of the Grade 9 pupils who took part in the initial pilot project seeks to illustrate this point.

Vignette

Grant was presented with the two non-consecutive terms shown in Figure 3. In the space of one hour he managed to determine nine different visually-mediated expressions for the nth term of the pattern. This vignette describes a meta-analysis of the 3½ minutes he spent arriving at the expression $n+n+n+[n-1]$.

Grant began by counting the forward-leaning parallel matches of Shape 5 from left to right. After a brief pause he then worked his way back from right to left counting the backward-leaning parallel matches. He then counted the remaining top and bottom matches in pairs, rhythmically alternating between top and bottom: 1,2…3,4…5,6…7,8…9. This rhythmic counting procedure, and its associated inherent sense of expectancy, was central in alerting Grant to the non-paired match in the bottom row.

Grant was thus able to arrive at the general expression $n+n+2n-1$. He justified this expression by relating the $n+n$ portion to two sets of "parallel central matches", while the $2n-1$ he associated with what he referred to as the "outside matches". Here we see usage of words such as "central" and "outside" which are indicative of a distinct spatial location and serve as spatial deictics. In addition, the word "parallel" functions as an important structural descriptor. Just prior to writing the $2n-1$ part of his expression, Grant gestured a horizontal line across the top of Shape 5 and a second horizontal line across the bottom of Shape 5. He also made the comment that "it'll always be one less on top", use of the word "always" performing a generative action function and thus aiding the notion of generality.

Interestingly, there seems to be a slight miss-match between the $2n-1$ portion of Grant's expression and his indexical gesturing of the top and bottom rows of matches in Shape 5 – the "outside matches". Upon further interrogation it was revealed that Grant saw the structure in terms of n matches along the bottom and $n-1$ matches along the top, and the √ portion of his expression was in fact an algebraic simplification of $2n-1$. Grant went further to describe the $[n-1]$ as representing the "top gap-filling matches", a metaphorical visualisation of the spaces created between the inverted V-shapes formed by the two central series of parallel matches. Grant then re-wrote his expression for the nth term as $n+n+n+[n-1]$.

There seems to be an interesting tension between two different modes of operative apprehension as evidenced by Grant's semiotic means of objectification of his general expression. Figural modification has been accomplished by means of a recombination of various elementary figural units in two different ways. Although Grant ultimately presented the expression $n+n+n+[n-1]$ as being representative of his visual perception of the figural pattern under investigation, his initial formula was √. Although he maintained that he had written ç as an algebraic simplification of $n+[n-1]$, the $2n-1$ may well have been unconsciously inspired by his original counting procedure in which the rhythmical pairing of the top and bottom matches was central in alerting him to the non-paired match in the bottom row.

Concluding Comments

It was the purpose of this paper to establish a research framework for an investigation into the extent to which pupils are able to visualise figural cues (objects with both spatial properties and conceptual qualities) in multiple ways within the context of pattern generalisation. Enactivism, along with the construct of knowledge objectification, were identified as forming an ideal theoretical framework for such a study, while the notion of figural apprehension proved central in elucidating visual tension. An initial pilot study revealed that some pupils have a surprising facility to visualise figural cues in multiple ways, while a meta-analysis of research participants' semiotic activity proved particularly revealing in terms of the process of visualisation.

The cognitive significance of the body has become one of the major topics in current psychology. Furthermore, the use of multiple representations has been acknowledged as playing a central role in problem solving, the learning and understanding of mathematical ideas, and the development of a deeper appreciation for the interconnections between mathematical concepts. By focusing on issues of visualisation and pattern generalisation, central components of mathematical activity, this study adds to an important pedagogical discourse by engaging with the critical notion of mathematical accessibility. Hamilton (2006) comments that, "… learning refers to transformations that expand the learner's potential range of actions" (p. 4). Pedagogical insights gleaned from the broader study seek ultimately to empower pupils with appropriate strategies to interpret figural patterns in multiple ways by moving flexibly between different modes of apprehension, thus creating the potential for just such transformations.

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