Abstracting by Constructing and Revising a 'Partially Correct Construct': A Case Study

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This study draws on data from a broader video-stimulated interview study of the role of optimism in collaborative problem solving. It examines the activity of a Grade 5 student, Tom, whose initial constructing activity resulted in a 'Partially Correct Construct'. Insistent questioning from another group member pressuring for clarification led to Tom developing a 'more correct construct' with further potential for revision. This paper raises questions about influences that can stimulate or inhibit construct refinement.

As early as the nineteen-seventies, researchers had begun to focus on the development of deep mathematical understanding (Krutetskii, 1976 in Williams, 2007a), and the need for teaching more than just rules and procedures to enable this to occur (Skemp, 1976). It was soon recognised that working with unfamiliar challenging problems and discussing ideas with other students (Wood & Yackel, 1990) supported the development of these deep understandings. This study examines how a student working with an unfamiliar challenging problem in a classroom in which student-student interactions were encouraged refined their understandings over time, and the influences that supported this change.

Theoretical Framework

The 'abstracting' of new knowledge can occur when a student or group of students interact to explore a mathematical complexity that was not evident to them at the commencement of a problem solving task, and they spontaneously decide to explore it (Williams, 2007a, 2007b). Abstracting involves student/s creatively 'building-with (B)' (Dreyfus, Hershkowitz, & Schwarz, 2001) previously known ideas after 'recognizing (R)' their relevance. Through synthesis they illuminate something mathematically profound. This is a process of 'constructing (C)' new knowledge. During the process of abstracting new knowledge, which includes processes R, B and C, students 'consolidate' (Co) their understandings. This involves recognising this knowledge in other contexts, and using it more flexibly. Abstracting in context (Schwarz, Dreyfus, & Hershkowitz, 2009) includes the following activities:

> 'Reorganisation within mathematics, … finding shortcuts and discovering connections between concepts and strategies (pp. 17)
>
> 'Explanations underwent a transformation that appeared to support … reaching a mathematically valuable understanding' (pp. 16).
>
> Prior constructs … are … reorganised … [and] ideally, … also integrated and interwoven (pp. 17).
>
> Key: '…' text omitted without change meaning; '[Text]' Elaboration of researcher.

In other words, students during the process of abstracting can assemble mathematics they recognize (R) as useful for a given purpose, find new ways to combine this mathematics (novel B), consolidate (Co) their new understandings during the process of using them for further exploration, and integrate mathematical ideas they develop to gain mathematical insights ('constructing', C) (e.g., Williams, 2007a; 2007b).

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The Engaged to Learn pedagogical approach (Williams, 2007b) used in this study is expected to provide opportunities for students to develop deep understanding. It is based on students working together on ‘conceptual tasks’ (Lampert, 2001) at small group and whole class level. Study of students’ learning through this approach has shown it does elicit frequent creative activity during the development of new conceptual understandings (see for example, Barnes, 2000). The strength of the Engaged to Learn Model lies in the accessibility of the tasks through a variety of mathematical pathways, and the enabling of group autonomy to control the difficulty of the mathematics they choose to explore; within a teacher-set focus. Different groups tend to approach the task in a variety of ways.

The process of abstracting has been found to begin with the formation of an amorphous entity that gradually gains internal and external structure during mathematical exploration (Hershkowitz, Schwarz, & Dreyfus, 2001; Williams, 2007a). The construction of ‘Partially Correct Constructs’ (PaCCs) as part of the abstracting process has recently been a focus of attention. These PaCCs are students’ ‘knowledge construct[s] that only partially matches a mathematical knowledge element that underlies the learning context’ (Ron, Dreyfus, & Hershkowitz, 2009, p. 1). PaCCs may develop through students not recognizing boundary conditions within which the new construct is relevant. This study extends that focus by examining how a PaCC changed over time, and what influenced the change to a ‘more correct construct’?

Research Design

This section includes the task, the context in which it was implemented, and the data collection instruments and why they were appropriate to this study.

The Fours Task

Use four of the digit 4, and any number of the following

+ + - - x ÷ / ( ) √ 2 .

to make each of the whole numbers from 1-20.
Then look for ways to find them all as fast as you can. Explain.

This was the third task in a sequence of three tasks undertaken across the school year in an upper elementary school classroom in Melbourne. The task was undertaken in one eighty-minute session. The researcher implemented the tasks and team taught with the teacher. Both intended to ask questions to help students to clarify their ideas, and extend their thinking, but not to direct, affirm, or query the pathways students took, nor provide mathematical input during student work with the task. This was for the purpose of enabling student autonomy. It is not always easy to achieve as this study shows.

What makes this task complex, and likely to lead to the creative development of new mathematical ideas, is that groups are asked to do more than find answers. They are asked to think about, and report on, thought processes they used as they tried to find integers, and to develop systems to find multiple answers. They were also encouraged to develop ‘big ideas’ to help find integers fast. Students undertook the task individually for the first three minutes, and then shared their ideas with other group members as a start to group work on the task. Table 1 shows the cyclic nature of the lesson structure [Column 2] within the Engaged to Learn approach. Intervals are described in Column 3. The majority of the time was spent on group reporting [Column 1]. After Alf’s report [Interval 5], Tom’s ideas began to develop [Column 3, Interval 5]. He refined them when Gabrielle persistently required further explanation and specific examples [Column 3, Intervals 7].

Table 1
Activity During Fours Task, Time on Each Activity, and Change in Tom's Understandings

Time	Interval No. / Title	Interval Description
9 Mins	1. Task Introduction	Find all the whole numbers from 1-20 using a restricted number of the digit four and given operations.
3 Mins	2. Individual Work	Students worked silently for three minutes starting task.
8 Mins	3. First Group Brain-storming	Shared ideas, found more numbers, thought about processes used, looked for fast ways
4 Mins	4. First Priming of Reporter	Group decided what to share. Reporter communicated this to group who refined the report.
30 Mins	5. First Group Reports to Class	Reports (1-2 Mins) included: something causing difficulty, specific examples, and / or strategies used. Alf's report provided cognitive artifact for Tom.
3 Mins	6. Refocusing Groups	The Researcher-Teacher (RT) refocused groups on thought processes, elegance, ways to generate numbers, and identifying big ideas to develop strategies.
5 Mins	7. Second Group Brainstorming	T spent some time directing Tom's group. Gabrielle then pressured Tom for explanations/ specific examples.
4.5 Mins	8. Second Priming of Reporter	Tom, as reporter communicated his intended report to his group, consolidated his understandings, and progressed towards a 'more correct construct'.
16.5 Mins	9. Second Reports	Tom reported explicitly about changed in understanding

The composition of Tom's group was decided upon by the RT who had opportunity to analyse video of students working in groups on the previous two tasks. Groups contained 3-4 students with similar paces of thinking, and a student who was likely to be able to keep their group on task and encourage all students to participate (Gabrielle in Tom's group). Tom was interviewed after the Fours Task.

Classroom video and video stimulated post-lesson student interviews were employed to study the process of creative development of new knowledge. Four video cameras were used to capture the six groups and the reporting sessions. There were audio leads from each group. The videos were mixed as 'two ups' so Tom could simultaneously see the activity in his group, and the reports at the board. Tom controlled the drag function on the video to select the intervals in the lesson that he wanted to view and discuss. This video stimulus assisted Tom to reconstruct his thinking in class, what influenced this thinking, and how he was feeling during these intervals. Tom's expressed feelings of high positive affect were used to help identify a situation in which he gained new insight (Barnes, 2000). These two data sources together provided a chronology of changes to Tom's conceptual understanding, and progressive influences upon it.

Results and Analysis

In Interval 3 (see Table 1), Tom built with his previous knowledge as he used trial and error (B) to find an integer '… eight, four plus four minus four plus four …' (16:05), and then consolidated his recall of the plus 4 minus 4 sequence by using it again several times

(B, Co) in similar contexts '… Four times four, plus four minus four, is 16!! Got it!' (19:18), 'Four times four plus four minus four' (19:55), and 'Four times four is sixteen, minus four is twelve. Plus four is sixteen. I made sixteen!' (20:23). Tom gave no indication that he was aware he was using a similar structure in each of these expressions. He made the same integer twice (16) and seemed equally surprised each time. He recognised (R) traces of B in another context when Helen used the same type of structure with the +4 and -4 in the opposite order 'That's just what I did in a different order' (29:35). Even with this consolidation, unlike some other students working on this task (e.g., see Williams, 2007b), he did not appear to recognise -4+4 and +4-4 as a mathematical object 'zero' nor that the two operations on the same digit cancelled each other out.

Tables 1 and 2 include the visual image of the process of construction, and social influences upon it developed by Dreyfus, Hershkowitz, and Schwarz (2001). It includes the time, and transcript line number in the lesson [Column 1], the 'observable cognitive elements' of the process of abstracting for Tom (R, B, C, Co) that Tom was undertaking at that time [Column 2], and social influences on this process [Column 3]. It also includes a summary of transcript excerpts [Table 2, Column 4] or the transcript excerpts [Table 3]. Table 2 shows activity during the first reporting session and second group work session [Interval 5, Interval 7, Table 1] that was relevant to Tom's exploratory activity. The transcript for Table 2 is presented below:

238. [36:22] Zeb [To class] … four plus four is eight … four divided by four is one … plus them together because there is a plus between four plus four and four divided by four.

279. [46:29] Alf [To class] … seventeen … we did four times four to get sixteen but we needed one more … we had two extra fours … then we did four times four plus … four over four … so it would be like saying four times four plus one …

283. [47:41] Alf [To class] Umm, well four over four is one whole, so that is just like saying one, and four times four plus one you get seventeen.

341. [1:01:03] Tom [to group] … we need … a strategy to figure out every single one … it could be … like what Alf and Ken's group did because four over four… one could come in handy for everything that is a not multiple of four- so … from sixteen you need one to get to seventeen ... umm- something minus four over four to get to fifteen.

344. [1:01:09] Teacher (T) [to Tom] … maybe do you want to give an example, do you think that would be better?

345. [1:01:42] Tom [to T] Maybe.

346. [1:01:53] T Sounds good to me- you might just want to give an example- sometimes examples really help yeah?

347. [1:01:58] Tom: so I think you could go- I don't know if you could count that as one [Alf's one]- to make that into a minus of that - so to get to sixteen, you go four times four umm. Err, oh yeah! Then you could go minus four over four, and yeah that is fifteen.

348. [1:02:35] T So that was one part of your argument. What was the second part of it?

348a. Tom [Looks unsure]

350. [1:03:04] T So- which ones [multiples]? Is this four over four going to work for?

351. [1:03:09] Tom Oh, I don't think it does. Because it is either one more or one less than.

352. [1:03:16] T: [Helen yawns] Okay, I'm just getting the impression from your three partners that body language anyhow- that they are not fully following you. So you might have to try some examples.

Table 2

Tom's Social and Cognitive Activity Associated with New Ideas He Started to Develop

Line/Time	C	B	R	Co	Tom A Z T	Description of Transcript
L238 36:22						Zeb reports to class using four on four as one in specific case
L279 46:49						Alf reports to class identifying four over four as a mathematical object (one) added to sixteen
L283 47:41				√	Ex	Alf reinforces what he has just said
L341 1:01:03			P		El	Tom extends Alf's idea to using plus or minus one as a stem following a multiple of four
L344 1:01:48					C	Teacher (T) suggests Tom give an example
L345 1:01:52 L346 1:01:53					At	Tom's response suggests this is not what he wanted to do
						T affirms Tom's direction and again suggests example
L347 1:01:58				√	Ag C	Tom explains his ideas the same way again without further examples
L348 1:02:35					E l	T asks for the second part of Tom's argument which he has not elaborated yet
L348a					C Q	Tom looks unsure
L350 1:03:04						T: "So, which ones? Is this four over four going to work for?"
L351 1:03:09					C Q	Tom: "Oh, I don't think it does. Because it is either one more or one less than"
L352 1:03:16						T draws attention to the lack of engagement of other group members and again requests examples.

Key: R: Recognizing, B: Building-with, C: Constructing, Co; Consolidating; P, Partially correct recognising
A: Agreement, Q: Query, El: Elaboration, Ex: Explanation, C: Control, At; Attention

Tom did not recognize the relevance of Zab's report [Table 2, Transcript Line 238] but Tom recognized the usefulness of 4/4 in Alf's report. Tom excitedly gestured by twirling his hand around in the air during the interview as he explained:

> … when he [Alf] said four over four and it is the same as one just that sentence just flung me like quickly in my mind: ahhh I could use that.

Why did Tom recognize the relevance of Alf's report and not Zab's? Was it because Alf was considered to be outstanding at mathematics by the class and the teacher? Or was it because Zab reported B where Alf reported as a 'big idea' or newly constructed object: that 4/4 could be used as one [Table 2, Transcript Lines 279, 283]. Or was it some combination of these possibilities? Tom began to develop a system for finding multiple integers by using Alf's new entity [Table 2, Transcript Line 351]. His insight: he could use plus or minus 4/4 after a 'stem' made using the other two digit 4s. This was the correct part of his Partially Correct Construct. He thought the stem would be any multiple of four between 1 and 20. This was the part that was not yet correct. Tom knew the plus or minus 4/4 would give the integer on either side of the stem made with the other two fours, but not that it was not possible to make all multiples of four between 1 and 20 using the + 4 – 4 sequence he had consolidated during the first group work session (described earlier). His initial thinking that he would be able to do so became apparent in his interview:

> I tried that one [making 12] just at the last minute as I was walking up [to report] … you would have to do four plus four plus four to equal 12 and then you can't do four plus four because that has two fours in it so that is five fours which is three fours which gives 5 fours in all

Tom had begun to realise that it was not necessarily easy to make multiples of four using two of the digit four but had not yet realised 12 and some other integers could be made in other ways using two of the digit four (without the + 4 – 4 sequence).

Thus, there is potential for Tom to extend his construct in relation to possible stems. At present, the boundaries for what is possible using his 'more correct construct' are too narrow. Tom knew the plus or minus 4/4 would give the integer on either side of the stem made with the other two fours, but not that it was not possible to make all multiples of four between 1 and 20 using the + 4 – 4 sequence he had consolidated during the first group work session (described earlier). His initial thinking that he would be able to do so became apparent in his interview:

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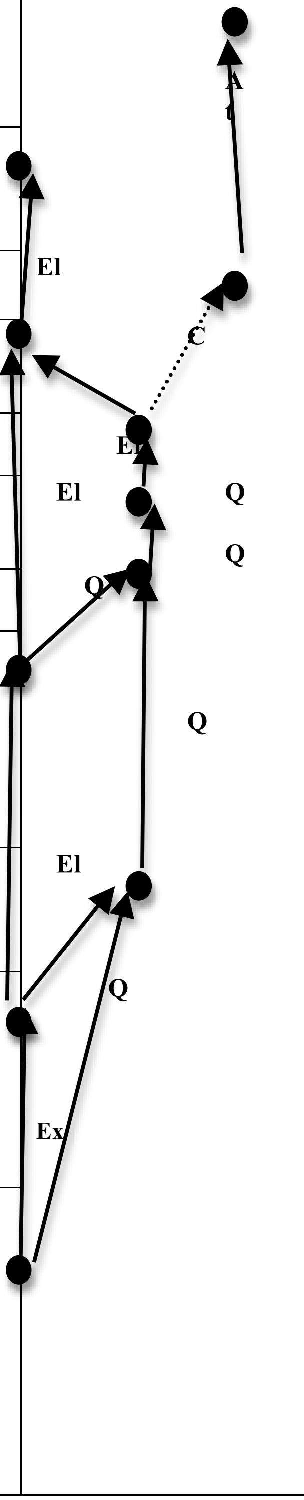
At Line 344 in Table 2, Tom's constructing process was interrupted before he had time to think further about whether all multiples of 4 could be built (B) as stems. Table 2 shows the absence of further constructing by Tom [see Table 2, Column 2] as he responded to the teacher questions that controlled what he thought about. This is also represented by the absence of arrows pointing back to Tom's previous responses as part of what influenced his later responses. Once the teacher observed and commented on the disinterested body language of the other group members [Table 2, Transcript Line 352], the teacher encouraged the students to interact by requesting Gabrielle prove Tom's theory [Table 3, Transcript Line 355]. With the teacher having validated Tom's idea, and the rest of the group now paying attention, Tom reiterated his theory several times in different ways without justifying why he considered multiples of four could be made as the stem each time [Fig. 2. Transcript Lines 353, 357, 364].

Table 3 shows the social elements of the interaction changed significantly once the teacher left and Gabrielle began to pressure insistently (or query, Q) for explanation of why Tom considered multiple of four could be made: 'How? How? How?' [Table 3, Transcript Line 360]. The arrows point back from Tom's responses to Gabrielle's queries and to Tom's own previous responses showing Tom was progressively building on his previous thinking. Column 2 shows where Tom's thinking involved R B and C thus representing the new constructs developing. When Tom kept elaborating (El) on the correct part of his construct (**B, Co**), rather than explain why he considered he could make multiple of four in the stem (**C**, idea not yet fully developed), Gabrielle focused her queries more directly on the aspect of Tom's ideas she did not understand: 'Use the four and four' [Line 363]. Gabrielle wanted to know why Tom thought he could make all multiples of four using two of the digit four. When Tom continued focusing on the plus or minus 4/4, Gabrielle pressed for specific examples of how each number could be made [Line 365]. It was this insistent 'querying' to get further explanation that led to Tom finally realising he could not make 12 in the way he had expected he would be able to (see quote above).

Table 3

Gabrielle's Intense Query: Tom Elaborates, Further Explains, Realises and Corrects Ideas

Line/Time	Cognitive Elements				**Social Elements**	Excerpts of transcript associated with Tom's progress from partially incorrect to more correct construct
	C	B	R	Co	**Tom Gabrielle T**	
Line 352 1:03:16						T: [Helen yawns] Okay, I'm just getting the impression from your three partners that body language anyhow, that they are not fully following you. So you might have to try some examples.
Line 353. 01:03:27	P					Tom: [Group listen, Helen fiddles] So for four you can get 3 and 5- using four over four- and for eight you can get 6 and 7- for 12 you can get 13 and 11- for 16 you can get 15 and 17- and for 20- you can get 19
Line 355 1:03:50						T: Yeah- come on Gabrielle- so get started- he has got a pretty good theory- and now you'll have to prove it.
Line 357 1:03:58			P			Tom: [7 secs] [Recording numbers] And then there are all of these ones- Four- Eight- Twelve- Sixteen- Twenty. [Looks up at group] Guys?
Line 358 1:04:29						Gabrielle: [No response from group. Gabrielle and Tom stare at each other.] (...) go over it and tell
Line 360 1:04:33						Gabrielle: [Takes pencil from Tom, shifts sheet to middle of table, and taps pencil on page as speaks] *How*? *How*? *How*? [group lean in towards page as she speaks]
Line 363. 1:04:40						Gabrielle: [Tapped finger on page requesting more about how the stem works] Use the four and four.
Line 364 1:04:41			P	√		Tom: Using four over four- that means one- … four quarters is one … or a whole- so that means that using a whole- you can either go minus four over four- which means one- so minus one- or plus one- so we did that- and these are all multiples of four- 18, 12, 8, 4. And then using minus four over four- or plus four over four- you can get these numbers- 17, 15, 13, 11, 7, 6, 5, 4, 3
Line 365 1:05:21						Gabrielle: [Takes pencil and paper] Hang on- so if … somebody asked you to make- to give answers to every single number- ones that you could possibly get with using a whole
Line 375 1:07:25				√ P		Tom: [practising during priming reporter time] Er, okay. Alf and Ken said- that using four over four- as they said was one- so using minus that- which would be the same minus one- and also plus one- you can get anything which is in a range of one number of multiples of four. So if it was 12- the numbers are like 11 and 13. And using four over four you can get okay, umm
Line 377 378 390 1:08:20						Tom: [writing on Gabrielle's table in priming time] So an example would be four times four minus four over four- which would be fifteen- so four times four is sixteen, and minus- that is just like saying minus one- so that is fifteen. But, if you use- if you are going to do like twelve- you won't be able to do it- because four plus four- plus four is one of the only ways to get to twelve … so the only one you can do it for is sixteen … oh unless you do the eight- so four plus four- yeah, so for eight- so four plus four is eight- minus four over four

Key: As for Table 2

Discussion and Conclusions

After developing a PaCC by starting to see an elegant 'short cut' that used the structure of the expressions he made to make more than one integer, Tom's '[e]xplanations underwent a transformation that appeared to support … reaching a mathematically valuable understanding' (pp. 16, see previously). Gabrielle's persistence in getting Tom to explain the part she was not understanding (the incorrect part of the PaCC) led to Tom revising his construct to one which was no longer incorrect but did not yet show recognition of the potential for other stems. Tom had boundaries for the use of his construct that were narrower than what was possible. There were some other multiples of four he could build, and there could be other possibilities that were not multiples of four.

An interesting question that remains is the role played by the teacher intervention. Did the teacher legitimise what Tom was doing to the extent that the rest of his group were prepared to listen to him? Or did the teacher's intervention (that eliminated autonomous thinking) interrupt Tom's constructing and delay him gaining his group's attention? Did the teachers comment to Gabrielle [Table 3, Line 355] lead to her insistent questioning of Tom? Or would this have occurred anyway? Previous interactions of Gabrielle's in other groups would suggest this type of questioning was part of her usual interactions. Further case studies are needed to learn more about PaCCs and whether they change over time to become more correct constructs, and what influences these changes. The present study shows such changes can occur, and aspects of the Engaged to Learn approach can facilitate such change: the group interactions helped to highlight what was not yet justified, and the reporting process contributed cognitive artefacts and helped to crystallise understandings.

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