Algebraic Thinking: A Problem Solving Approach

Will Windsor
Griffith University
<w.windsor@griffith.edu.au>

Algebraic thinking is a crucial and fundamental element of mathematical thinking and reasoning. It initially involves recognising patterns and general mathematical relationships among numbers, objects and geometric shapes. This paper will highlight how the ability to think algebraically might support a deeper and more useful knowledge, not only of algebra, but the thinking required to successfully use mathematics. The paper will highlight how a deeper analysis of mathematical problems can instigate student discourse, providing meaningful experiences that can developing algebraic thinking.

Recent calls for reform in mathematics education in Australia have focused on the need to promote and facilitate improved teaching and learning of mathematics (Council of Australian Governments, 2008; National Curriculum Board, 2008). A key element of this reform agenda is the introduction of a national mathematics curriculum. The strand Number and Algebra will be an integral part of the new curriculum with the middle and upper primary years emphasising an algebraic perspective of number rather than the formal algebra familiar to most people. In contrast, secondary school students will undertake the study of formal algebra having been introduced to algebraic concepts and ideas at the primary school level.

The algebraic perspective I attempt to illustrate is a perspective that values, enriches and improves the thinking required to understand algebraic concepts. Consequently, in this discussion we will adopt the term 'algebraic thinking' rather than an algebraic perspective of number, as it takes into account the variety of activities students can engage in across all mathematics strands, not simply number. Algebraic thinking is founded on numeracy and computational proficiency, the reasoning of geometry and skills associated with measurement- concepts introduced and taught in the primary and middle school (Kaput, 2008). Importantly, it extends the thinking required to solve problems beyond methods tied to concrete situations. In time, this thinking supports students ability to problem solve using abstractions and to operate on mathematical entities logically and independently from the material world.

The approach that I propose involves identifying and using mathematical problems that promote and advance a generalised perspective of mathematical problem solving. An ability to consider problems from this perspective can allow individuals to acquire adaptable ways of thinking, to express the generalisations they have arrived at and leads into a meaningful use of algebraic symbolism (Carraher, Brizuela, & Schliemann, 2003). The potential value for using problem solving contexts is that it may broaden and develop students' mathematical thinking and provide them with an impetus for understanding a greater collection of problems of increasing complexity and mathematical abstraction (Kaput, 2008; Kaput, Blanton, & Moreno, 2008; Schliemann, Carraher, & Brizuela, 2007; Lins, Rojano, Bell, & Sutherland, 2001). As will become apparent, algebraic thinking promotes a particular way of interpreting mathematics. It extends the mathematical thinking of students by encouraging them to interact and engage with the generalities and relationships inherent in mathematics. Lins et al (2001, p. 3) contend that "no matter how suggestively algebraic a problem seems to be, it is not until the solver actually engages in its solution that the nature of the thinking comes to life."

L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia.* Fremantle: MERGA.

Problem Solving and Algebraic Thinking

Using mathematical problems has been advocated as a crucial and motivating component of learning and understanding mathematics. Schoenfeld (1992) notes that, when solving mathematical problems, students develop a deeper understanding of mathematics because it helps them to conceptualise the mathematics being learnt. Stanic and Kilpatrick's (1989) review of problem solving indicates that historically, mathematical problem solving has been instrumental in achieving a variety of goals within the mathematics curriculum. Furthermore, productive problem solving experiences that move children beyond the routine acquisition of isolated techniques are fundamental in developing higher order mathematical thinking and reasoning (Booker & Bond, 2009; Polya, 1973). In my view, many of the fundamental ideas on which mathematics is built can make sense to children if those concepts are viewed in meaningful and challenging contexts.

In the book *How to Solve It* (1973) Polya is explicit in characterising the heuristics of effective problem solving. Essentially, he attempts to understand how people think and the strategies they might use when solving problems. Polya (1973) contends that to solve any problem, the characteristics and properties of the problem should be analysed. Once the problem is understood then a plan is devised and strategies are implemented and finally, opportunities to reflect upon the solution are required. Though Polya emphasises the heuristics of problem solving, he also acknowledges the idea of mathematical connectedness and generality, key components of algebraic thinking. He suggests that by actively engaging with problems students can develop the ability to understand the generalities associated with problem solving:

> In solving a problem of one or the other kind, we have to rely on our experience with similar problems and we often ask the questions: Have we seen this problem in a slightly different form? Do we know a related problem? (Polya, 1973, p. 151)

This brief overview of Polya's work serves to emphasise that the generalities, relationships and interconnectedness that underpins mathematics can be carried through to developing algebraic thinking. Furthermore, being aware of and having the capacity to consider, ascertain and communicate the generalities of a particular problem may invariably enhance an understanding of formal algebra. As Krutetskii (1976, pp. 334-335) observed "one must be able to see a similar situation (where to apply it), and one must master the generalised type of a solution, the generalised scheme of a proof or of an argument (what to apply)." This perspective appreciates that algebraic thinking and problem solving are inextricably linked by common skills and mathematical understandings.

In shifting the emphasis of problem solving, from simply finding a specific answer to also including a focus on algebraic thinking, I conjecture that it may provide a powerful way to teach and learn algebraic ideas. Many of the problems found in elementary arithmetic and geometry have the capacity to support a way of thinking that connects a range of mathematical content and processes (Booker, Bond, Sparrow, & Swan, 2010; Bednarz & Janvier, 1996). Extending mathematical problems solving to include the developing algebraic thinking, educators can facilitate more divergent and adaptive ways of thinking mathematically. Opportunities arise to engage and extend students' mathematical experiences that go beyond routine arithmetical solutions. As Silver, Ghousseini, Gosen, Charalambous, and Strawhun (2005, p. 288) observed, when discussing the advantages of facilitating a variety of different solutions:

An aphorism of unknown origins captures the essence of this idea: “You can learn more from solving one problem in many different ways than you can from solving many different problems, each in only one way”.

Developing Algebraic Thinking Using Problem Solving

Teaching algebraic thinking using a problem solving approach can be established amid the learning experiences that already exist in most classrooms. It is apparent that this approach evolves and builds upon a child’s ability to consider, see and think about the mathematical concepts within a problem. Lee’s (2001) analysis of algebraic thinking highlights some of the underlying strategies which characterise this type of mathematical reasoning. She observes that when children analyse problems from an algebraic thinking perspective they may consider:

- Reasoning about patterns (in graphs, number patterns, shapes, etc) stressing and ignoring, detecting sameness and difference, repletion and order.
- Generalising or thinking in terms of the general, seeing the general in the particular;
- Mentally handling the as-yet-unknown, inverting and reversing operations;
- Thinking about mathematical relations rather than mathematical objects.

It is apparent that the development of algebraic thinking arises from generalising mathematical thought. Researchers such as Bednarz, Kieran, and Lee (1996) extend this idea and state “the process of generalisation as an approach to algebra appears ultimately related to that of justification”. Thus, a classroom environment that values and promotes collaborative learning situations, student discourse and the opportunities to communicate mathematical ideas and conjectures can better facilitate algebraic thinking. The resounding importance of teachers to facilitate algebraic thinking through meaningful discourse can be observed in the research of Carpenter, Franke, and Levi (2003), Carraher, Schliemann, and Brizuela (2003) and van Amerom (2002). Each of these research teams encouraged student discourse so as to promote deeper mathematical reasoning and expedite algebraic thinking.

In focusing on developing algebraic thinking, current thinking suggests students will progress through three stages of development. At first, many students will describe generalities and relationships in natural language, this can lead into abbreviating those ideas by using diagrams and mathematical symbols and finally, these ideas can be summarised using mathematical expressions and equations, tables of values and graphs (NCTM, 2000; Mason, Graham, & Johnston-Wilder, 2005). The history of mathematics would also suggest that to understand and solve problems of an algebraic nature, individuals operate and constantly manoeuvre their thinking between the before-mentioned three stages often referred to as rhetorical, syncopated or symbolic stages (Harper, 1987; Katz & Barton, 2007). Clement, Lochhead, and Monk (1981) describe that most mathematicians think this way; rarely do they consider their thoughts in a purely symbolic realm. Instead, describing their ideas as being like pictures — with tables, graphs and symbols used to interpret those ‘pictures’. However, the thinking required to understand these thoughts is algebraic because there is a focus on the general rather than the specific.

The work of scholars such as Lannin, Barker, and Townsend (2006) and Booker and Bond (2009) serve to reinforce the importance of students interacting with the problem, their teachers and other students at a variety of different levels. Lannin, Barker, and Townsend describe how social factors, cognitive factors and task factors simultaneously influence the way students address problems and how this influences their reasoning. Booker and Bond (2009) document the effectiveness of working collaboratively and the value of encouraging multiple perspectives. As they both document, developing algebraic thinking can be achieved when students are encouraged to use a variety of strategies and are supported to communicate their ideas, reflect upon solutions and have opportunities to speculate about the concepts and ideas they have constructed.

Algebraic Thinking within a Classroom Context

This section will illustrate and highlight how the transition from the rhetorical to the syncopated stage of generalised thought can be promoted. The following three problems were completed by a Year Six class. The tasks involved finding a pattern and possibly explaining a method for summing an arithmetic series. The students worked in seven groups of four and were able to use a calculator, pencil and paper or counters to come with a solution or explanation.

Problem A - Chiming Clock

An old chime clock strikes one chime at 1 o'clock, two chimes at 2 o'clock, three chimes at 3 o'clock and so on. How many chimes will it strike in a 12-hour cycle?

Problem B - Counting Coins

Your New Year's resolution is to save enough money to buy a new bike. You decide to put $1 away on the first day of the year, $2 on the second day, $3 on the third day, $4 on the fourth day and so on. How much money will you have after 30 days?

Problem C - A King's Ransom

A King told his knights that if they could slay the dragon they would be richly rewarded. He informed them that he would place on a chess board one gold coin on the first square, two gold coins on the second square, three gold coins on the third square, four gold coins on the fourth square and so on. How many gold coins will the knights have if they slay the dragon? There are 64 squares on a chess board.

At first many children simply added successive numbers. However, many had difficulty in explaining or justifying their solutions in generalised terms. Using their calculators or pencil and paper many children could achieve the correct solution for Problem A by simply adding the numbers in the correct sequence. As the children attempted to understand and solve the other two problems it became obvious that this method was inefficient and cumbersome. This point can be illustrated by the fact that when the groups proceeded to the second question five of the seven groups each had a different answer even though their solution strategies were very similar. Through the ensuing discussion there came the realisation that Problems B and C were similar to Problem A but there must be a more efficient way to solve the Counting Coins and A Kings Ransom problems. One child explained this to her group as follows:

Ashley: There has to be a better way (and takes the calculator). It takes us too long to do the other two problems. There's 'gotta' be a pattern. It's like what Mr H showed us the other day with Pascal's Triangle.

At this point, the children were encouraged to use counters or a diagram to explore the first problem again. (See Figure 2). One of the groups who used a table commented that it reminded them of their rainbow facts.

Peter: *This is like the rainbow facts we did in grade one.*
James: (Laughing) *They all add to 13.*
Teacher: *All of them?*
James*:* *Yeah see.1 and 12, 2 and 11, 3 and 10.* (Runs his fingers over the connecting lines) *See they all add up to 13.*
Peter: *You add them?*
Teacher: *Is there a better way than simply adding 13 each time?*
Peter: (Starts tapping his pencil beside each 13).
James: (Looking at the teacher). *Is it the same as multiplication?*
Peter: (Picks up his calculator and enters 13 x 6) *Seventy-eight. It's the same as our answer.*

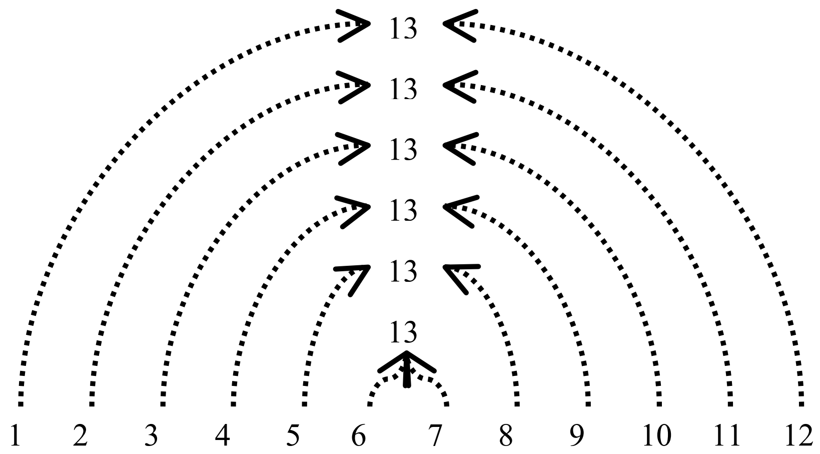

Figure 2. Chiming Clock solution similar to the "Peter and James" case.

As the lesson continued, three groups used counters to illustrate the Chiming Clock problem. (See Figure 3.) Each group represented a chime as a counter, with one counter representing 1 o'clock; two counters 2 o'clock and so on. The teacher moved to a group as a member from another group also watched the following episode.

Teacher: (Moves the single counter in the first row to the last row.
Then moves the next two counters to the second last row).
Annie: (Moves the next three counters into third last row, then the next four counters and so on). *There's six rows of 13 so 78 chimes. So all you do is add the numbers and multiply by the number* (pauses) *of pairs*.
Teacher: *Does it work for the other problems?*

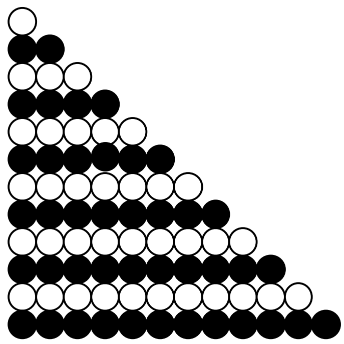

Figure 3. Using counters to interpret the Chiming Clock problem.

Annie and her group were then observed experimenting with other sequences. Importantly, the teacher let Annie and her group continue their exploration. Her group spent the remaining 25 minutes making different arithmetical sequences and comparing their solutions with a calculator. (See Figure 4.) As her teacher was walking past to attend to another group Annie commented, "And all you need to do is the same for all of the problems?" Her teacher acknowledged this statement with an approving nod.

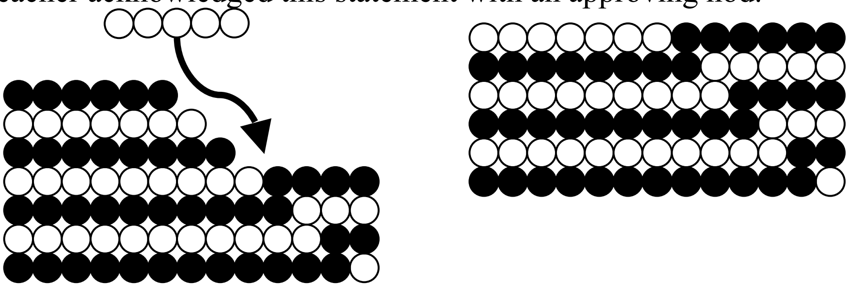

Figure 4. An example of Annie's thinking using the counters.

I infer from Annie's explanation and her group's use of the counters that the multiplicative structures of the problem were more apparent to groups who decided to use the materials. Using the counters suggested to the children an arrays model for multiplication that was more easily understood by this class. In the next phase of the lesson, her group discussed the Counting Coins and A Kings Ransom problems. Consequently they were also able to explain their solution algebraically and in general terms at a rhetorical level, moving into the syncopated phase. The group inferred that for the Counting Coins problem and A Kings Ransom problem that the same ideas existed in each of the problems respectively. (See Figure 5).

Teacher:	*How did you find a solution so quickly? Come up to the front and show us.* (points to an overhead projector and hands Paul a pen).
Paul:	*Well, we knew the numbers added to 31 and divided 30 by two. So it was 31 multiplied by 15 which is $465* (He writes the number sentences on the overhead projector).
Teacher:	*Can you do it with the other problems?*
Paul:	*They're all the same. You can solve them the same way.*

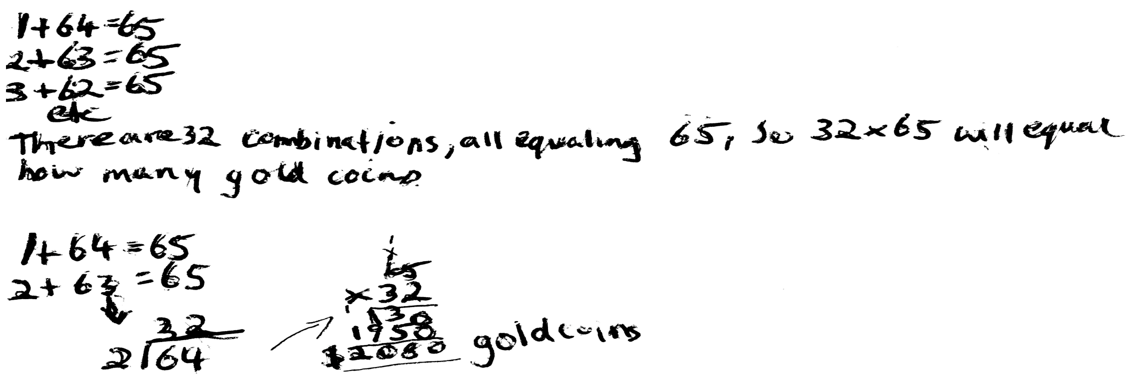

Figure 5. An excerpt from a student's work book.

The lesson concluded with the teacher asking "What would happen if we wanted to sum successive even numbers or successive odd numbers? What would happen if we started at a number other than one or there was not an even number of values?" The ability for the class to use the generalisations they had developed during the lesson created opportunities to shift their thinking from a purely answer focused perspective of mathematics. The sample episodes show an increasing sophistication in the way students worked through the problems. In the course of the lessons, the students explained and justified their responses to each other and were more productive in their capacity to develop generalisations about the problems.

Discussion and Conclusion

Developing algebraic thinking using a problem solving approach may build upon and extend the teaching practices used within many classrooms. However, it may compel some teachers to see problem solving from a different perspective. At a minimum it entails seeing problem solving as an opportunity to enrich and transform students' thinking rather than the 'ferreting out' of an answer. One of the distinctive characteristics of this approach is that it requires teachers to adapt and change the problems, yet maintain the mathematical generalisations present within the problems. Through appropriate discourse teachers can encourage students to think algebraically rather than influencing them to use a particular strategy or procedure. It is through discussion during the solving process that ideas relating to algebraic thinking and an algebraic perspective of mathematics can be developed. Encouraging students to reflect on their thinking and share their experiences can assist in students developing different ways of thinking about problems. As Silver et al (2005, p. 13) observes:

> The presentation of multiple solutions and the consideration of connections between and among different approaches to a problem could be seen as opportunities to advance the mathematical agenda.

A central challenge in addressing reforms in algebra can be addressed at many levels. As Thomas quietly commented at the end of his class discussion, "This is like that algebra stuff they will teach us at high school. It's not really different to what we do at the moment." If the themes I have identified for developing algebraic thinking address only students' perceptions of algebra in a positive manner than the merits of advocating a problem solving approach should warrant further investigation. Moreover, the benefit for developing students' algebraic thinking beyond the mechanics and procedures often associated with algebra can possibly offer students a more complex and meaningful conceptualisation of algebra. Using a problem solving approach to develop algebraic thinking and providing an algebraic perspective of mathematics may enhance the long-term learning trajectory of the majority of students.

References

Bednarz, N. & Janvier, B. (1996). Emergence and development of algebra as a problem solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran & L. Lee (Eds.) *Approaches to algebra: Perspectives for research and teaching* (pp. 115-136). Dordrecht, NL: Kluwer Academic Publishers.

Booker, G., Bond, D., Sparrow, L. & Swan, P. (2010). *Teaching primary mathematics (4th ed.).* Sydney, NSW: Pearson Education Australia.

Booker, G. & Bond D. (2009). *Problem-solving in mathematics.* Perth, WA: RIC Publications.

Carpenter, T. P., Loef Franke, M. & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary.* Portsmouth, NH: Heinemann.

Carraher, D. W., Schliemann, A. D. & Brizuela, B. M. (2003). *Bringing out the algebraic character of arithmetic: From children's ideas to classroom practice* [Electronic version]. Mahwah, NJ: Lawrence Erlbaum Associates. Retrieved 5 May 2009, from http://www.earlyalgebra.terc.edu/publications.htm

Clement, J. Lochhead, J. & Monk, G. S. (1981). Translation difficulties in learning mathematics. *American Mathematical Monthly, 88, 286-290.*

Harper, E. (1987). Ghosts of Diophantus. *Educational Studies in Mathematics 18*, 75-90.

Human Capital Working Group, Council of Australian Governments (2008). *National numeracy review report: May 2008* [Electronic version] Retrieved 8 November 2008 from *www.coag.gov.au/reports/docs/national_numeracy_review.pdf*

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). New York: Lawrence Erlbaum Associates.

Lannin, J., Barker, D. & Townsend, B. (2006). Algebraic generalisation strategies: Factors influencing student strategy selection. *Mathematics Education Research Journal, 18*(3)*,* 3-28.

Lee, L. (2001). Early algebra – But which algebra? In H. Chick, K. Stacey, J. Vincent & J Vincent (Eds.), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI Study Conference, pp. 392 – 399). Melbourne, Australia: University of Melbourne.

Lins, R., Rojano, T., Bell, A. & Sutherland, R. (2001). Approaches to algebra. In R. Sutherland, T. Rojano, A. Bell & R. Lins (Eds.), *Perspectives on school algebra* (pp. 1-11). Dordrecht, NL,: Kluwer Academic Publishers.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235-272). New York: Lawrence Erlbaum Associates.

Kaput, J. J., Blanton, M. J. & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19-55). New York: Lawrence Erlbaum Associates.

Katz, V. and B. Barton (2007). Stages in the history of algebra with implications for teaching. *Educational Studies in Mathematics 66 (2)*, pp.185-201.

Krutetski, V. A. (1976). *The psychology of mathematical abilities in schoolchildren* (J. Teller, Trans) Chicago, IL: University of Chicago Press.

Mason, J., Graham, A. & Johnston-Wilder, S. (2005). *Developing thinking in algebra.* London, UK: Paul Chapman Publishing.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

National Curriculum Board. (2008). *National mathematics curriculum: Framing paper.* Retrieved 8 December 2008, from *www.ncb.org.au*

Polya, G. (1973). *How to solve it: A new aspect of mathematical method 2nd edition.* Princeton, NJ: Princeton University Press.

Schliemann, A. D., Carraher, D. W. & Brizuela, B. M. (2007). *Bringing out the algebraic character of arithmetic: From children's ideas to the classroom.* Mahwah, N. J.: Lawrence Erlbaum.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C. & Strawhun, B.T.F. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal for Mathematical Behavior*, *24*, 287-301.

Stanic, G. & Kilpatrick, J. (1989). Historical perspectives on problem solving in the mathematics curriculum. In R. Charles & E. Silver (Eds.) *The teaching and assessing of mathematical problem solving* (pp. 1-22). Reston, VA: National Council of the Teachers of Mathematics.

van Amerom, B. A. (2002). *Re-invention of early algebra: Developmental research on the transition from arithmetic to algebra.* Unpublished doctoral thesis, Netherlands: University of Utrecht.