Equivalent Fractions: Developing a Pathway of Students' Acquisition of Knowledge and Understanding

Monica Wong
University of Sydney
monica.wong@optusnet.com.au

Learning pathways capture the development of competence in a mathematical domain. They have been developed from empirical studies in the areas of mental computation and emergent numeracy concepts. These pathways afford teachers the opportunity to identify students' current levels of understanding, antecedent understandings and the steps that are likely to result in students achieving a more sophisticated level of understanding. A pathway of the skills and knowledge that students acquire in developing conceptual understanding of fraction equivalence was developed through the assessment of 649 students from Grades 3 to 6 attending six primary schools. The assessment, analysis of data and hypothesised pathway for area models are described in this paper.

Many students are unable to construct or identify equivalent fractions (Bana, Farrell, & McIntosh, 1997; Pearn, 2003; Siemon, Virgona, & Corneille, 2001). Ni (2001) laments that understanding of fraction equivalence is often reduced to "mastery of the rule 'multiply or divide the numerator and denominator of a fraction by the same number'" (p. 413), with many students resorting to the application of a memorised rule or inventing their own.

To advance students' learning with understanding, teachers must gain insight into the paths students follow in developing understanding of fraction equivalence. Learning pathways are one method of capturing students' development of competence in a subject-matter domain. They are considered an evidence-based model of learning which depicts students "as starting out with little or no knowledge in the domain and through instruction gradually building a larger and larger knowledge base" (National Research Council [NRC], 2001b, p. 182). Such developmental models can be used to identify students' current levels of understanding, antecedent understandings, and the steps that are likely to result in students achieving a more sophisticated level of understanding (NRC, 2001b).

This paper describes a section of a larger three-phase study. The first phase comprised the development of a pencil and paper instrument, the *Assessment of Fraction Understanding (AFU)*, which was used to measure students' conceptual understanding of fraction equivalence (Wong, 2009). During Phase Two, *AFU version 1* was administered to 297 students. After analysis of the results, the instrument was reviewed and revised, resulting in the creation of *AFU version 2*, which was administered to another 349 students during Phase Three. The process of developing a pathway to capture one potential route to understanding fraction equivalence using area models, which incorporates the quantitative components of Phases Two and Three, is discussed in this paper.

Theoretical Perspective

Learning Pathways

Learning pathways have been developed from empirical studies in many areas of mathematics. Some studies such as Siemon et al. (2001) examined numeracy with particular focus on assessing students' knowledge of "key, underpinning mathematical ideas and their capacity to apply and communication this knowledge in context" (p. 6).

L. Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education: Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia.* Fremantle: MERGA.

This was achieved by assessing students using a pencil and paper test incorporating open-ended written questions. The use of Rasch analysis enabled the development of an eight-level numeracy pathway, which included "rich descriptions of distinct developmental levels" (Siemon et al., 2001, p. 29).

Students' mental computational competence in the area of fractions, decimals and percentages was assessed by Callingham and Watson (2004). Like Siemon et al. (2001), Callingham and Watson used a pencil and paper assessment and employed Rasch analysis to develop their *Levels of Mental Computational Competence*. Questions/items of comparable difficulty and cognitive demand were grouped together to describe possible levels of competence. Students with the lowest level of competence could answer only the easiest items, while only more competent students were able to complete the most difficult tasks.

Learning pathways have also been developed to support emergent numeracy learning (Mulligan, Looveer, & Busatto, 2006). However, an evidence-based pathway for understanding fraction equivalence is missing. The development of such a pathway will enable the identification of students' knowledge of fraction equivalence and their misconceptions, thus providing a platform on which teachers can base their lessons to improve students' understanding.

Equivalent Fraction Domain

A review of fraction literature (e.g., Cathcart, Pothier, Vance, & Bezuk, 2006; Lamon, 2005; NRC, 2001a) reveals that knowledge and understanding of fraction equivalence encompasses more than the procedure of multiplying or dividing the numerator and denominator of a fraction by the same number. Students with conceptual understanding of fraction equivalence have an integrated knowledge and are able to display and articulate the following five attributes.

1. A fraction represents a quantity being measured in relation to a referent unit.
2. A fraction quantity can be represented using manipulatives or pictorially by partitioning area, collection or number-line models.
3. Equivalent fractions can be constructed from manipulatives or pictorial representations by repartitioning or chunking.
4. Equivalent fractions can be constructed using symbolic notation.
5. A fraction quantity is a member of an equivalence class in which all fraction numerals represent the same quantity.

Students can represent this mathematical knowledge in various ways, using representations which comprise some of, or all five interrelated elements of spoken language, written language, manipulatives, pictures and real world situations (Lesh, Landau, & Hamilton, 1983). Examples of the different representational elements for the fraction one-quarter are depicted in Figure 1. Students who possess conceptual understanding know when and how these representations can be used for different purposes. They are able to co-ordinate links or map from one representation to another (e.g., pictorial to symbolic) and within representations (e.g., area to number-line diagrams).

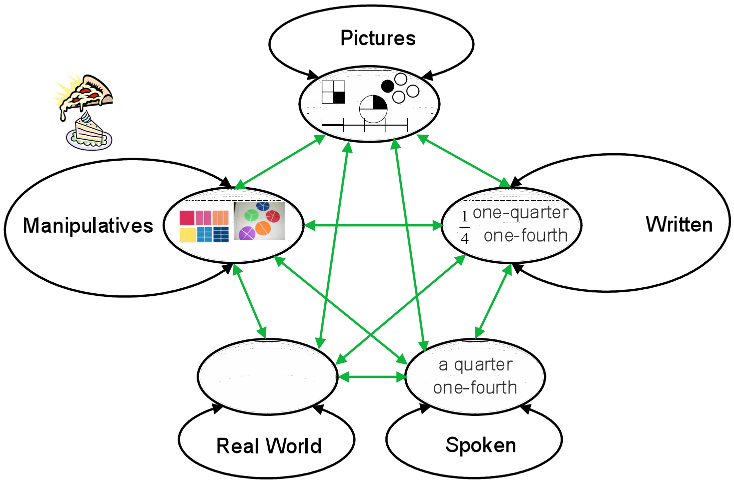


Figure 1. Examples of representational elements for the fraction one-quarter.

Assessing Students' Conceptual Understanding of Fraction Equivalence

For teachers to verify when learning has occurred, students must be provided with opportunities to demonstrate what they have learnt. Hence, the *Assessment of Fraction Understanding* (AFU), a pencil and paper assessment incorporating tasks, which addressed the five fundamental aspects of conceptual understanding, was developed (Wong, 2009). It was used to assess students' conceptual understanding of fraction equivalence, using pictures and written language. These data enabled the exploration of the following research question: "What learning pathway do students travel on their journey to development of conceptual understanding of fraction equivalence incorporating area models?" In formulating a model of students' development of knowledge and understanding of fraction equivalence, the National Research Council offers important advice:

> There is no single way in which knowledge is represented by competent performers, and there is no single path to competence. But some paths are travelled more than others. When large samples of learners are studied, a few predominant patterns tend to emerge. (NRC, 2001b, p. 182)

Stacey and Steinle (2006) suggest that pathways reflect teaching practices, while Moseley (2005) advises that students' exposure to fraction perspectives is influenced by the curriculum. Irrespective of these concerns, it is identification of a pathway that the majority of students travel towards conceptual understanding of fraction equivalence that is the focus of this paper.

The Study

Participants, Data Collection and Instruments

Data collection was conducted across Phase Two and Phase Three as depicted in Figure 2. Six hundred and forty-six students in Grades 3 to 6 attending six co-educational urban primary schools (three Catholic and three government) participated in the study. During Phase Two, all students were administered the *Assessment of Fraction Understanding version 1* (AFUv1), which comprised 31 constructed-response questions, some with multiple parts, which resulted in 47 items for Rasch analysis (Wong, 2009). All instruments were reviewed by mathematics educators, instrument designers and other researchers during development.

The results of Phase Two data analysis informed the construction of *Assessment of Fraction Understanding version 2* (AFUv2), as shown in Figure 2. Unreliable items from *AFUv1*, those identified with layout inconsistencies, issues with wording and clarity of instruction, were removed or reworded and more items added to create *AFUv2*. *Form A* comprised 25 questions or 32 items, while *Form B* comprised 27 questions or 35 items. Twenty-five items were common across *Form A* and *Form B*, of which 16 were retained from *AFUv1*. Each instrument is detailed in Wong (2009). During Phase Three, students were administered either *Form A* or *Form B* depending on their grade level, as shown in Figure 2. All assessments were administered following standardised protocols. Participants were asked to work independently and were allowed 45 minutes to complete the assessment.

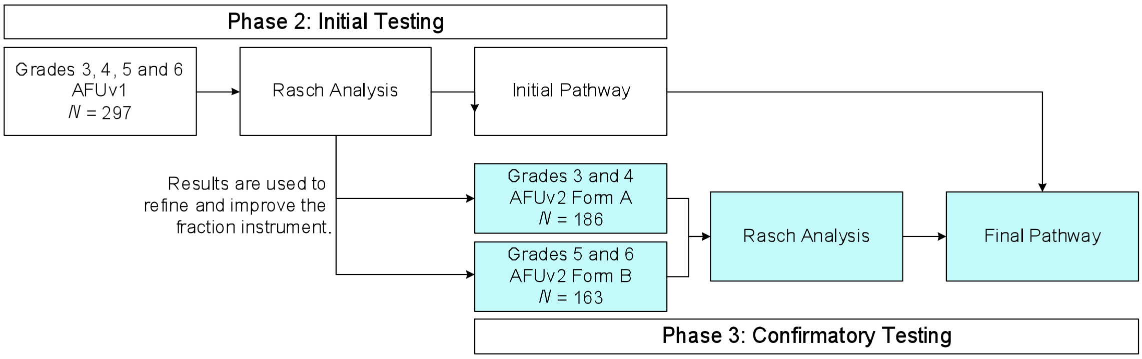


Figure 2. The quantitative data collection and analysis employed in phases two and three of the study.

Rasch Analysis

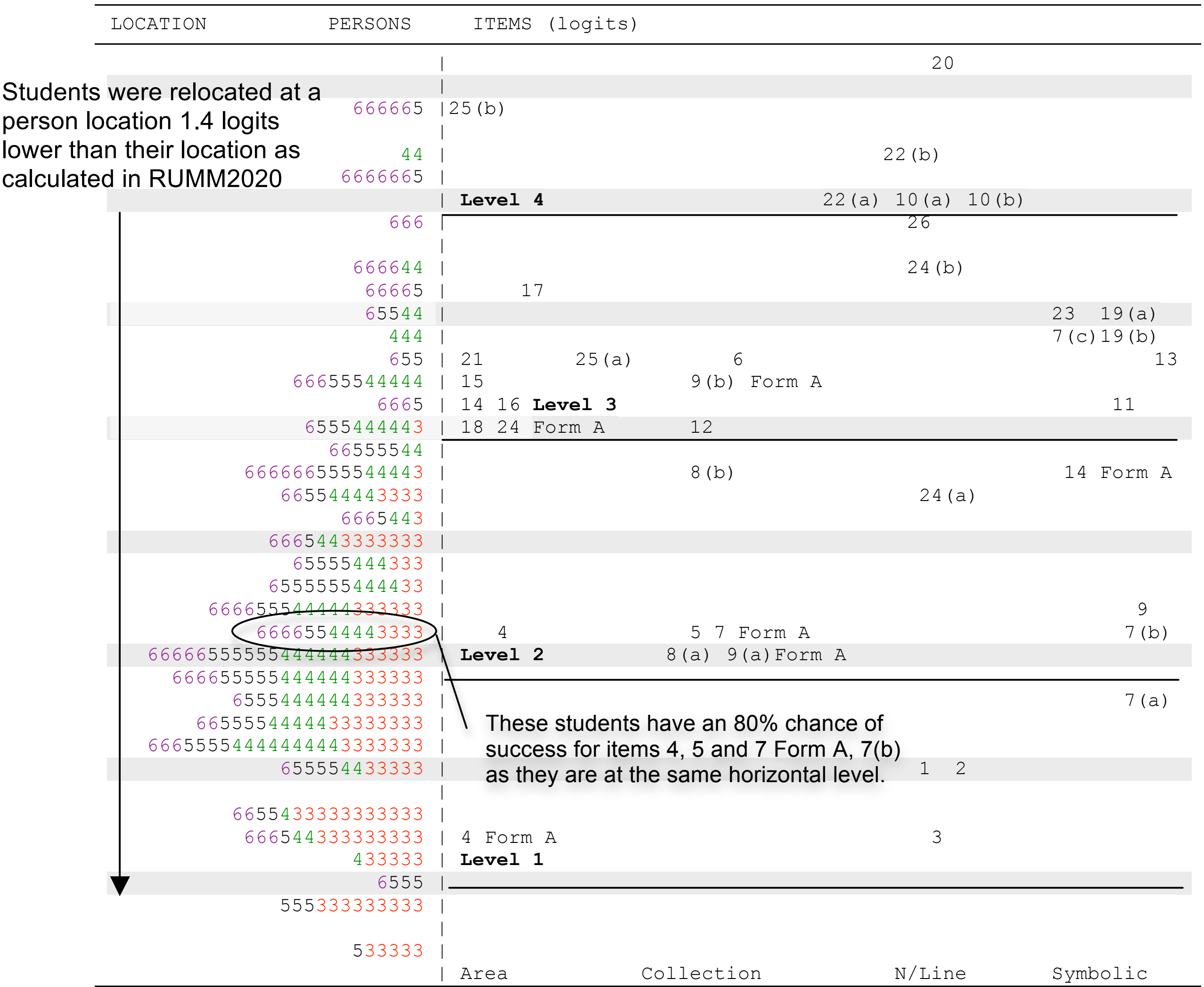
One key feature of Rasch analysis is that students and the questions or items they attempt can be placed on a common scale (Wright & Stone, 1979). Students from a range of grades can be assessed without the need for all students to be administered all items, as instruments can be designed with common items to allow comparison of students across grades (Wright & Stone, 1979). A set of common items was included across *AFUv1* and *AFUv2* Forms A and B.

Using RUMM2020, the difficulty of each item in the *AFU* was estimated, along with person location or students' conceptual understanding of fraction equivalence on a common logit (log-odds) interval scale. Firstly, the data collected from initial instrument testing were analysed and an initial pathway of understanding was developed. This was followed by a revision of the equivalent fraction instrument, *AFUv2*. Another 349 students were assessed using the *AFUv2* during confirmatory testing. The pathway of understanding was verified and updated with the results of the Rasch analysis from confirmatory testing. A discussion of the final pathway follows.

Results and Discussion

The person-item map produced by RUMM2020 indicates that a person whose person location or trait level matches the difficulty of an item (same horizontal location), has a 50% probability of success on that item. However, Bond and Fox (2007) suggest that an 80% probability of success on an item represents mastery level learning, hence the measure of students' knowledge and understanding in the domain. Therefore, the RUMM2020 person-item map was adjusted by shifting each person's location by 1.4 logits downwards. Thus students and items at the same level represent an 80% chance of success by the student on that item. For example, the students circled in Figure 3, possess an 80% chance of success for item 4, 5, 7 *Form A* and 7(b). Items (e.g., 14, 15 and 25) above their level are more difficult, whilst items (e.g., 1, 2 and 3) below are easier.

From the person-item map of Figure 3, a hierarchy of student attainment and conceptual understanding of fraction equivalence tasks was derived. Items within the person-item map were arranged by fraction model – area, collection, number-line or symbolic notation. Four possible levels of understanding were distinguished, as shown by the horizontal lines. These boundaries were determined by grouping items of similar difficulty and features (e.g., fraction size – less than one, unity or greater than one). The lower bound for level 1 was positioned below item 4 Form A (see Table 1 for item description) as rudimentary understanding of fractions incorporates the identification of area representations for one-half (Callingham & Watson, 2004). Below this level, students were unlikely, to consistently answer, even the easiest item correctly.



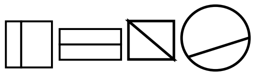
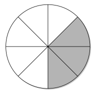
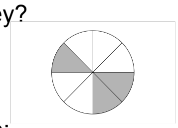
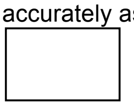
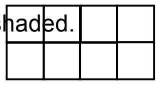
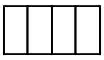
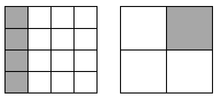
Each 3, 4, 5 or 6 represents one person in his/her respective grade.

Figure 3. Excerpt of person-item map for AFUv2 showing mastery levels of understanding.

The pathway of understanding exhibited by students participating in this study is described in Table 1. A description of the skills and sample items from the assessment are included. Although the area pathway is only described in the following section, other pathways exist for number line and collection models (Wong, 2009). While the pathway is presented as a linear progression, students can reside across multiple levels within a pathway.

Table 1

Levels of Understanding of Fraction Equivalence using Area Models

Skill	Example Items
Level 1 (raw score 2 - 8)	
Recognise the quantity half, represented in a simple area model.	4 (Form A). Circle the shapes that have been divided in half. 
Level 2 (raw score 9 - 21)	
Recognise a fraction quantity represented in a simple area model.	4. What fraction has been shaded grey?  Alternative: 
Level 3 (raw score 22 - 30)	
Represent a fraction quantity by partitioning an area model.	18. As accurately as possible, shade $\frac{2}{8}$ of the rectangle. 
Represent a fraction quantity using an equivalent representation.	14. In the rectangle, shade enough small squares so that $\frac{3}{4}$ of the rectangle is shaded.  15. Shade in $\frac{2}{2}$ of the shape below. 
Recognise the fraction quantity represented.	24 (Form A). Has the same fraction of each large square been shaded?  Explain how you know?
Level 4 (raw score 31 - 40)	
Recognise the fraction quantity represented.	25.  This rectangle represents one whole. (a) What do the following represent altogether? (b) Can you think of name for the fraction shaded?

Teachers can monitor and measure students' knowledge and understanding of fraction equivalence by administering either AFUv2 Form A or Form B, without the need to examine students' responses to individual items. Each level of mathematical understanding is described in terms of a raw score, which is calculated by summing the number of correct responses from the assessment. Using a student's raw score, the student can be located within a level of conceptual understanding of fraction equivalence. If the raw score falls at, or near, the boundary between levels, the student may be able to complete some of the tasks above their descriptor level, but not consistently. Students possess approximately an 80% probability of being able to perform the skills identified at their particular level.

The first level pertains to recognising representations of the fraction one-half. Hart (1981) reported children "appear to recognise equivalents of a half and to deal with them in a different way to other fractions" (p. 76). It does not necessitate students to link the notion of half to one of two equal pieces with Hayward and Fraser (2003) confirming that students with limited understanding refer to any part of a whole, irrespective of size as "half" or "quarter".

Students achieved understanding corresponding to Level 2, when they were able to identify a fraction represented by a simple representation (see Item 4, Table 1). A "double counting" process can be used to determine the fraction shaded, whereby, the shaded component represents the numerator and the total number of equi-sized parts represents the denominator. Students located below Level 2 on the person-item map of Figure 3, achieved a raw score of eight or less. These students lacked the knowledge that a fraction represents a relationship between a part, measured in relation to a whole, as they exhibited whole number counting by responding with the answer 'three' to Item 4.

Students achieved understanding consistent with Level 3, when they were able to partition an area model accurately. The majority of students partitioned the rectangle in Question 18 (see Table 1) into 8 equi-sized parts, shading two of them. The most prevalent errors incorporated partitions of unequal size. In some instances, students partitioned from one side, resulting in too few or too many partitions. For students with limited knowledge and understanding, inaccurate drawings thwarted their attempts at generating consistent and correct answers, similar to the findings of Hayward and Fraser (2003). Only 7% of students exhibited and applied their knowledge of fraction equivalence by converted $\frac{2}{8}$ to its equivalent $\frac{1}{4}$, and shaded one-quarter of the entire shape. These students possessed stable knowledge as they also completed Items 14 and 15 (as shown in Table 1) correctly.

Students located within Level 3 were also able to repartition a fraction quantity, using alternate equi-sized parts to derive a different fraction name (Lamon, 2005). These students were able to successfully shade $\frac{3}{4}$ of a shape divided into eight parts (see Item 14, Table 1). In contrast, students below this band exhibited limited understanding by shading only three small squares. A similar response was observed for the fraction $\frac{2}{2}$, where students shaded two parts. Thus, representing fractions using equivalent representations requires greater understanding and an increase in cognitive demand (Callingham & Watson, 2004), compared to items incorporating simple representations.

At the highest level of understanding, students were required to identify the fraction $1\frac{3}{4}$ and name an equivalent fraction (see Item 25, Table 1). Although the unit or whole was explicitly defined, students demonstrated unstable knowledge by combined both units to create a new composite unit (Vance, 1992), thus providing the response $\frac{7}{8}$. Few students were able to find an equivalent fraction for their response to Part (a). Hence, a quantity greater than one was more difficult for students to represent or recognise than those less than one.

Conclusion

The pathway identified from the study reflects one pattern of the development of conceptual understanding of fraction equivalence using area models, and captures the expectations of the *Mathematics K-6 Syllabus* (Board of Studies NSW, 2002). This pathway represents the knowledge gathered from Grade 3 to 6 students, attending six primary schools from two educational sectors. This diversity should reduce some of the effects of teaching practices and fraction perspectives promoted by individual schools and teachers. The pathway of understanding fraction equivalence should provide a valuable tool for teachers. Knowledge that students have mastered, and knowledge required to achieve a more sophisticated level of understanding, can be identified for any student who completes the *Assessment of Fraction Understanding version 2*. Therefore, implementation of the assessment and pathway in the classroom by teachers is the next step in verifying their accuracy and usefulness, which has been planned for future investigation.

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