The Four-Three-Four Model: Drawing on Partitioning, Equivalence, and Unit-Forming in a Quotient Sub-Construct Fraction Task

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This paper demonstrates the explanatory power of Kieren's framework for rational number knowing (1988, 1992, 1993, 1995), renamed here the four-three-four model, by describing the different approaches of Grade 6 students to a quotient context task (sharing three or seven custard tarts between five people) using Kieren's terminology of partitioning, equivalence, and unit-forming.

The explanatory power of Kieren's framework for rational number knowing, (Kieren, 1988, 1992, 1993, 1995) has rarely been examined on its own terms in the research literature on children's fraction understandings (see e.g., Mitchell, 2011). This is in contrast to the body of research that uses his five sub-constructs model (1980) consisting of part-whole, measure, quotient, operator, and ratio, which I will refer to as his five-part model. The Rational Number Project researchers (see e.g., Behr, Lesh, Post, & Silver, 1983) adapted his five-part model for their own research. Research on fractions in the Kieren/Rational Number Project research tradition overwhelmingly uses Kieren's five-part model (or minor adaptions thereof) to frame the categorisation of fraction tasks and fraction understanding (see e.g. Lamon, 2007; Pantziara & Philippou, 2012). Kieren's framework for rational number knowing (1988, 1992, 1993, 1995) refigured his earlier model and included *four* sub-constructs (measure, quotient, operator, ratio), *three* underlying concepts (partitioning, equivalence, and unit forming), other named constructs that made up the quotient field, and also a description of learning with *four* levels: ethnomathematic, intuitive, technical-symbolic, and axiomatic deductive. I have renamed Kieren's framework for rational number knowing the four-three-four model to distinguish it from the five-part model more commonly associated with his name.

This four-three-four model refigures the concept of part-whole, and as instruction based on a limited part-whole concept of fractions has been repeatedly shown to restrict children's development of a richer understanding of fractions (see e.g., Lamon 2007), the four-three-four model is worth another look. It would not be possible in one paper to examine the explanatory power of the whole four-three-four model, and so the model is not examined in its entirety here. In the present paper, I use part of the fraction construct aspect of the *four-three*-four model to analyse Grade 6 children's explanations of a quotient sub-construct task.

Review of the Literature

The Four-Three-Four Model

In Kieren's adaptation (1988) of his five-part model of rational number knowledge (1980), the four sub-constructs of measure, quotient, operator, and ratio remained, but were underpinned by three concepts: partitioning, equivalence, and unit-forming (see Figure 1). These sub-constructs and concepts could be engaged with on four levels of understanding. Kieren further elaborated this model in later writing (1992, 1993, 1995). The four-three-four model was not in opposition to Kieren's earlier five-part model (1980), nor to the model used in the Rational Number Project research (Behr et al., 1983; Behr, Harel, Post, & Lesh, 1992). Kieren saw the concepts of partitioning, equivalence and unit-forming as providing a

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better categorisation of the underlying fraction concepts that were described more obliquely in the work of the Rational Number Project (see e.g., Behr et al., 1992).

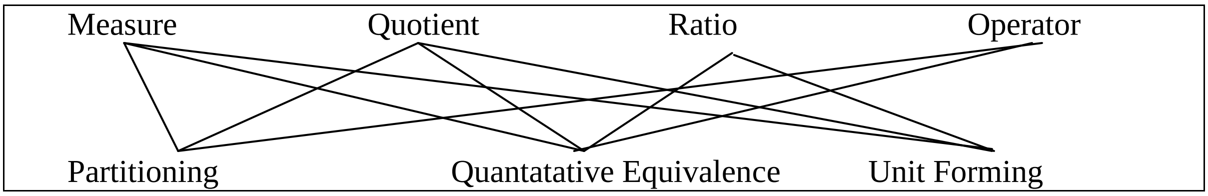


Figure 1. Section from Kieren's framework for rational number knowing (1993)

The description of learning in the four-three-*four* model has been further refined (see e.g., Pirie & Kieren, 1994), refiguring the four levels (ethnomathematic, intuitive, technical-symbolic, and axiomatic deductive) into eight layers. This aspect of the four-three-four model has been applied outside the fraction context and has stood on its own. The fraction construct aspect of the *four-three*-four model can also be applied independently from an analysis using the levels of understanding. The sub-constructs of Measure and Quotient are the only two in which Kieren suggests all three underlying concepts are being drawn upon (see Figure 1), and so I have sought to report on those two sub-constructs first. In my doctoral study I demonstrated the explanatory power of the four-three-four model with respect to primary school children's explanations of measure sub-construct tasks (Mitchell, 2011). The quotient sub-construct task described in the present paper was used in my doctoral study and is reported in more detail here. The present paper seeks to illustrate the usefulness of the four-three-four model in the description of explanations of quotient sub-construct tasks.

The Quotient Sub-construct

The quotient sub-construct described a context in which sharing between two separate measure spaces took place. For example, three pizzas shared between five people generated equal shares of three fifths: $3 \div 5 = {}^{3}/_{5.}$ Similar tasks have appeared in the research literature (Charles & Nason, 2000; Clarke, Roche, & Mitchell, 2011; Kieren, 1988; Lamon, 1999).

Children's strategies for attempting quotient tasks included the intuitive strategy of repeated halving: cut every pizza in half and deal out all the pieces and see if that works, if not, divide every pizza into quarters and deal out all the pieces, then repeat with eighths if quarters leave a remainder (Pothier & Sawada, 1983). If children then subdivided the remainder, their explanations sounded like "engineering reports" (Kieren, 1988). Children had trouble keeping track of the unit when doing non-equal partitioning (dividing each piece into half and then the leftover half into five), for example, sometimes calling the result "a half and a fifth" instead of a half and a tenth, or six tenths, or three fifths (Lamon, 1999). The Dutch curriculum introduced fractions with a sharing context and used the term French division to describe cutting each pizza into enough pieces for everyone (see e.g., Keijzer & Terwell, 2002); each of the three pizzas would be divided into five parts and a piece from each pizza dealt out to each person, resulting in three one-fifth shares. In a separate Australian study, just less than a third (30.3%) of Grade 6 students could solve the pizzas shared between five people problem (Clarke et al. 2011). Very few of them used the fractions as division concept that three shared between five was three over five or three fifths.

Partitioning

Kieren's description of partitioning was the folding and drawing actions required of children when making equal parts (1995). Stages have been elaborated for learning to partition including repeated halving and using the radius rather than the diameter of a circle to generate thirds and fifths (Pothier & Sawada, 1983).

Equivalence

The concept of equivalence had a specified place in the four-three-four model. In some other frameworks, equivalence understanding had been positioned as early ratio understanding or bundled as order and equivalence (Behr, Wachsmuth, Post, & Lesh, 1984), but it was understood to be an important fraction concept. A strength of the four-three-four model was that it gave equivalence an explicit place.

Unit forming

Unit-forming described the additive nature of fractions. Just as eight could be made of seven and one, or six and two, or five and three, so too fractions could be made from the sum of other non-equal fractional amounts (Kieren, 1995). Kieren called the instructional aspect of unit-forming the combining space and described classroom activities with concrete materials that were used to show fractions as "additively combinable amounts" (1995, p. 32). For example, students were offered the open ended problem: "here are four rectangular pizzas cut in halves, quarters, sixths, and twelfths. Choose some pieces from at least three of these pizzas such that their "sum" is one pizza" (Kieren, 1993).

Methodology

The results presented here were part of a larger study concerning children's explanations of fraction tasks and measurement tasks (Mitchell, 2011). 88 Grade 6 students from three schools in metropolitan Melbourne, took part in a one-to-one task-based interview and their responses to tasks were recorded on a record sheet, audio-taped, and over half were video-taped. Clinical interviews provide insight into children's explanations (Ginsberg, 1997) but cannot describe learning. Smaller studies, such as this one, if conducted meticulously, can provide credibility and insightful analysis, but the results are not generalisable.

The quotient sub-construct was assessed using a sharing situation similar to sharing tasks in the research literature that used pizzas and people (see e.g., Lamon,1999). Length and area representations were used, but pizzas were not chosen as the context because of the concern that children had preconceived ideas about the number of pieces into which they could be divided. The Sharing Custard Tarts or Liquorice task included four questions presented sequentially with a task card and figurines, and the students could use pen and paper to work out their solutions. All questions were spoken. Question A involved sharing three strips of liquorice between five people. Question B involved sharing three custard tarts between five people: “Five people are sharing three custard tarts equally. The tarts can be cut anywhere. How much of a tart does each person get?” Question C involved sharing seven custard tarts between five people: “Five people are sharing three custard tarts equally. The tarts can be cut anywhere. How much of a tart does each person get?” Question D involved sharing nine strips of liquorice between four people (see Figure 2).

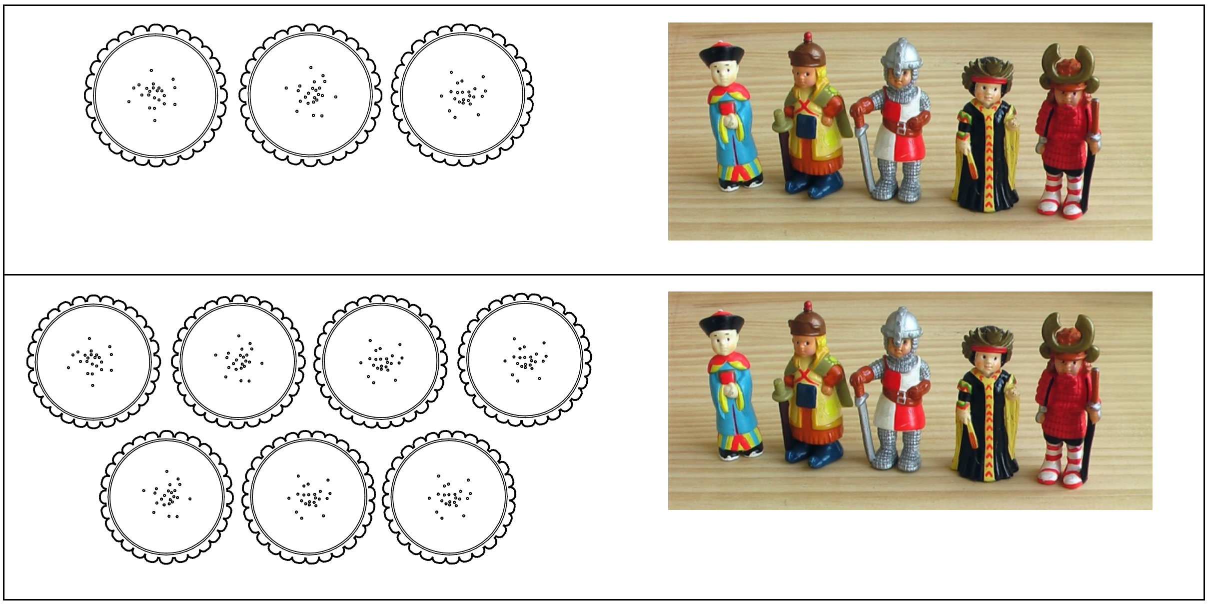


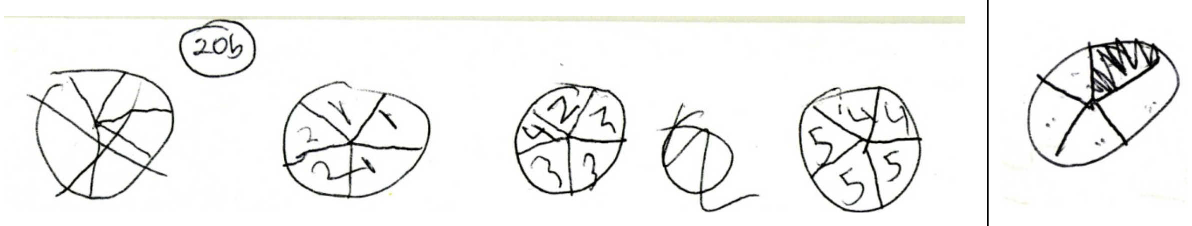
Figure 2. The Sharing Custard Tarts or Liquorice task: Question B and Question C.

Results and Discussion

Students' explanation to two of the questions, Question B (sharing three custard tarts between five people) and Question C (sharing seven custard tarts between five people), are examined here, with respect to how partitioning, equivalence and unit-forming are drawn upon in students' explanations of their answers. In Question B, 26.1% of students successfully made five equal shares (diagram or mentally) and named the share correctly as three fifths, while a further 36.4% could make five fair shares, either as fifths, or a half and a tenth, but were unsuccessful at naming the total share correctly. An improper fraction answer ($^7/_5$ or $1^2/_5$) was generated by the problem of five people sharing seven custard tarts (Question C) and 20.5% of the students could successfully share and name the share in this problem. A further 35.2% of the students could create or imagine equal shares but could not name one person's share successfully. In Question C, one student used a fraction as division approach, seven shared between five *is* seven fifths, and was operating at a level where he did not need to consciously recruit partitioning, equivalence or unit-forming concepts.

Partitioning

Partitioning was evident in the responses of the students to both questions. In Question B, some students used French division to share each custard tarts into five equal parts (see Figure 3). Meg, after a poorly executed first attempt at fifths (crossed out), drew three custard tarts, divided them into fifths, allocated shares, and named each person's share correctly as three fifths. Ebony (see Figure 3) was able to make the partitions and indicate a share, but named those three parts (one shown, the others visualised) incorrectly as three fifteenths.



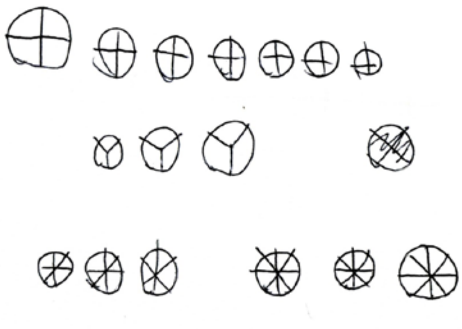
Meg: Three fifths

Ebony: $^{3}/_{15}$

Figure 3. Partitioning using French Division.

Similarly in Question C, some children used French division and divided all seven custard tarts into five parts and some successfully named one person's share as seven fifths or one and two fifths. Some children eliminated the wholes first and then divided each of the last two custard tarts into five, successfully naming this as one and two fifths. As in Question B, some children could do the partitioning but could not successfully name the share.

In Question B, Harry found the partitioning difficult with a circular area diagram. He divided a circle into six parts and after puzzling over more drawings he thought aloud, "there's no way you can cut it up into five pieces". Similarly, Brad (see Figure 4) tried partitioning the three custard tarts into thirds, eighths, and sixths (inaccurately) but could not find a partitioning that would enable fair sharing. For Question C, he partitioned the seven custard tarts into halves and then drew over that to partition them into quarters (see Figure 4). In both Question B and Question C, some children drew on partitioning and used repeated halving but could not create a number of parts that could be dealt out equally.



Brad: "Don’t know"

Figure 4. Unsuccessful partitioning in Question B.

Unit Forming

Unit forming was drawn upon in these two questions when students made fair shares by combining a number of non-equal parts. They often also recruited partitioning in the execution of their strategy. Some students who divided all five custard tarts in half then partitioned the leftover half into five and drew on unit-forming to name the combined fair share. Of the 19 students who did this in Question B and made five fair shares out of two unequal parts, only one could name this correctly. Seth drew unit-forming (making a fair shares of two unequal parts) while recruiting partitioning (dividing the leftover half into five equal parts), and then used equivalence ($^{1}/_{2} = {}^{5}/_{10}$) to name the share six tenths (see Figure 5).

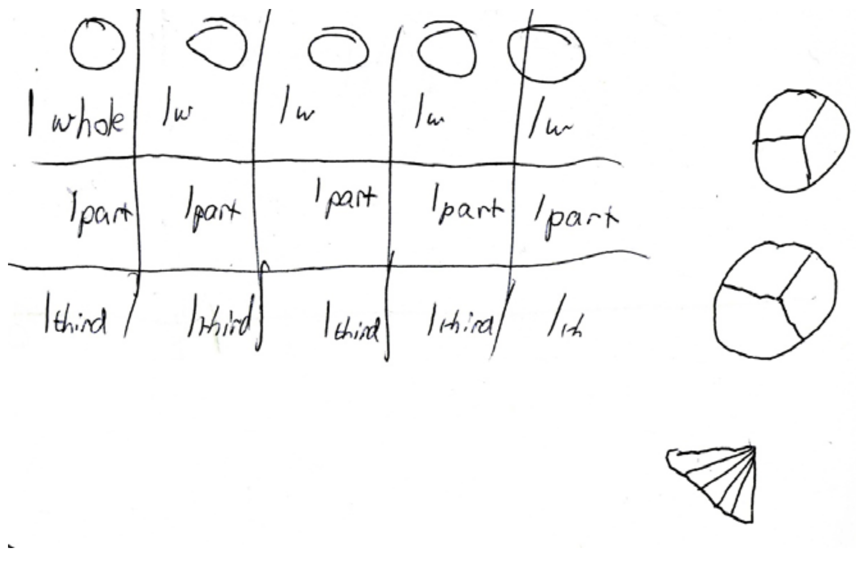
Seth: Six tenths

Figure 5. Successful unit forming in Question B.

Some children could describe the share, as a half and a fifth of the extra half, but did not have the equivalence skills to complete the renaming. Other children incorrectly described the share as a half and a fifth, unable to keep track of the unit.

On the other hand, one child, Felix, gave a reasonable estimate using unit-forming, but rather than recruiting equivalence knowledge, he visualising the two pieces combined and then drew on the concept of partitioning to estimate the size of that "part". He combined the half and one piece of the last part divided into five, and described this added together amount as “nearly two thirds”: this estimate (0.66) was very close to the exact value (0.6).

In Question C, some students drew on the concept of unit forming, for example, by eliminating the wholes, dividing the last two tarts into thirds, and then the leftover third into five parts. The equivalence knowledge needed to execute the naming of this share, made of combined amounts, was beyond the students who used this approach. However, one student, Courtney, was very logical with her representation (see Figure 6).



Courtney: one whole and one third and one fifth of the last piece

Figure 6. tracking unit-forming: sharing 7 tarts between 5 people.

One child, Ben, drawing on unit forming, made an estimate of his combined amounts, rather than calculate the fair share. After eliminating the wholes, he divided the last two custard tarts into halves. As this gave four parts, not five, he correctly reasoned that the fair share would be one and about three quarters of a half. His reasonable estimate of the non-whole number part, three quarters of a half ($^{3}/_{8}$ or $^{15}/_{40}$) was very close to the precise value of the part of one of the two leftover tarts ($^{2}/_{5}$ or $^{16}/_{40}$) that was needed to make a fair share when combined with a whole tart.

Equivalence

As described above, students who drew on the concept of unit forming needed to recruit equivalence knowledge to name the overall fair share successfully.

Conclusion

The present study demonstrated that the concepts of partitioning, equivalence, and unit-forming understandings were drawn upon by real students in different approaches to this quotient sub-construct task. The use of the terminology partitioning, equivalence, and unit-forming in describing the students' approaches to this quotient context has enabled a more detailed analysis of students' correct and incorrect responses to the task. The successful and unsuccessful responses described here that drew on partitioning, demonstrate that successful partitioning included imagining, making (diagram or mentally) and naming, sometimes non-contiguous, equal parts. Harry could imagine but not make partitions for fifths. Ebony could make but not name her partitioning when the parts were not contiguous. The results of the present study, particularly the infrequency of drawing on equivalence and the infrequency of using a fractions-as-division approach, demonstrate that the underlying concepts of partitioning, equivalence and unit-forming are still relevant in upper primary school.

A strength of the four-three-four model was its categorisation of unit-forming. This gives prominence to the many correct additive aspects of fraction understanding. Courtney's logical diagram (see Figure 6) is an example of Kieren's engineering reports (1988), and highlights that unit forming is a mathematically correct approach to quotient sub-construct fraction tasks that is not always fully executed. Using the concept of unit-forming, teachers and students could elaborate the restructuring of the whole in more sophisticated terms than a "part-whole" understanding in the five-part model allows. Similarly, the reasonableness of the estimates given by Felix and Ben, can be unpacked with greater depth in terms that more closely resemble their reasoning, by the use of the terminology of partitioning and unit forming. In Charles and Nason's study of Grade 3 children completing similar tasks, 12 strategies were reported but further categorisation under the term unit forming may have underscored the concept being drawn upon in their "repeated sizing strategy" (2000, p. 198).

The adoption of the four-three-four model would not be incompatible with the research already conducted using variations of the five-part model in the Kieren/Rational Number Project tradition. However, elaborating the categories of partitioning, equivalence (both already in use by teachers and students) and unit-forming would broaden teachers' pedagogical repertoire of part-whole instruction, as called for in the research literature on fractions. Equivalence has been a term in use with both teachers and students and so the concept has a recognised place in the curriculum. Extending the classroom vocabulary for both teachers and students, to include unit-forming would extend the explanatory power of the model to the students themselves.

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