Problem Posing in Mathematical Investigation

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This paper reports on the types of problems that high-achieving students posed when given investigative tasks that were constructed by opening up some mathematical problems. A teaching experiment was conducted to develop the students' thinking processes during mathematical investigation, and each student was videotaped thinking aloud during a pretest and a posttest. The findings show that some students were unable to pose the original intended problems and what Krutetskii (1976) called problems that 'naturally follow' from the task, including extending the task to generalise. The implications of the difficulty encountered by these students for teaching and research will also be discussed.

Background

Mathematical investigation is emphasised in many school curricula (e.g., Australian Curriculum, Assessment and Reporting Authority, 2012; Ministry of Education of New Zealand, 2007) because it is believed that investigation helps students to think and learn mathematically (Jaworski, 1994), and that investigation reflects the practices of mathematicians (Civil, 2002). Although there is plenty of research on problem solving (Cai, Mamona-Downs, & Weber, 2005), there are not many empirical studies on investigation despite a thorough search of current literature. Moreover, such research on investigation is usually restricted to examining the general benefits that investigation has offered to the students (e.g., Bailey, 2007; Tanner, 1989), but empirical studies on the types of processes that students actually engage in when doing investigation are rare. The study described in this paper addresses this 'gap' by investigating how students think during investigation. With a clearer understanding of the nature and development of these thinking processes, it might help teachers to cultivate these processes in their students.

Unlike mathematical problems, e.g., Task 1, that typically contain the question in the task statement, open investigative tasks, e.g., Task 2, usually do not contain any question but students are expected to pose their own problems to solve or investigate (Orton & Frobisher, 1996). Frobisher (1994) suggested that it is "nearly always possible to restate [a problem] in order to make it into an investigation" (p. 158). Using Task 1 as an illustration, Frobisher's idea would be to remove the *intended problem* from the task and replace it with the word 'investigate' (see Task 3), which he believed students would ultimately pose the intended problem and generalise by finding the shortest time to toast *n* slices of bread.

Task 1: Mathematical Problem (Toast)
Three slices of bread are to be toasted under a grill. The grill can hold exactly two slices. Only one side of each slice is toasted at a time. It takes 30 seconds to toast one side of a slice of bread, 5 seconds to put a slice in or to take a slice out, and 3 seconds to turn a slice over. What is the shortest time needed to toast the three slices of bread?

Task 2: Investigative Task (Square Each Digit and Add)
Choose any number. Square each digit of the number and add to obtain a new number. Repeat this process for the new number and so forth. Investigate.

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Task 3: Investigative Task (Toast)
Three slices of bread are to be toasted under a grill. The grill can hold exactly two slices. Only one side of each slice is toasted at a time. It takes 30 seconds to toast one side of a slice of bread, 5 seconds to put a slice in or to take a slice out, and 3 seconds to turn a slice over. Investigate.

The purpose of opening up a problem to an investigative task is to allow students to pose their own problems to solve. But the concern is whether the students know how to pose the intended problems. There also seems to be two different types of investigative tasks that tend to elicit somewhat different kinds of processes: Type A (e.g., Task 2) involves searching for any pattern by trying examples so as to generalise, while Type B (e.g., Task 3) requires students to *first* pose a problem like finding the time taken to toast the three slices and then solve it by using other heuristics such as deductive reasoning. It is beyond the scope of the paper to examine all the thinking processes that students engage in when performing both types of investigative tasks, so the paper will focus only on the problem-posing processes for Type B tasks. Hence, the research question for the paper is:

- What kinds of problems did the high-achieving students in this study pose when given Type B investigative tasks?

Literature Review

Thinking Processes

Clement (2000) observed that one of the most important needs in research on students' thinking processes is the need for insightful explanatory models of these processes, and that such research is most fruitfully undertaken using the thinking-aloud method (Duncker, 1926) and teaching experiments (Lesh & Kelly, 2000). Mason, Burton and Stacey (1985) described four main processes in mathematical thinking: specialising (or trying examples), conjecturing, justifying and generalising. Some researchers (e.g., Height, 1989; Cifarelli & Cai, 2004) have also designed a model to depict the thinking processes in investigation. Based on these models but with some modifications, an investigation model (see Fig. 1) was developed for this study to describe the interaction of these processes. An important difference between a mathematical investigation model and a problem-solving model is the additional phase of problem posing after understanding the task in investigation.

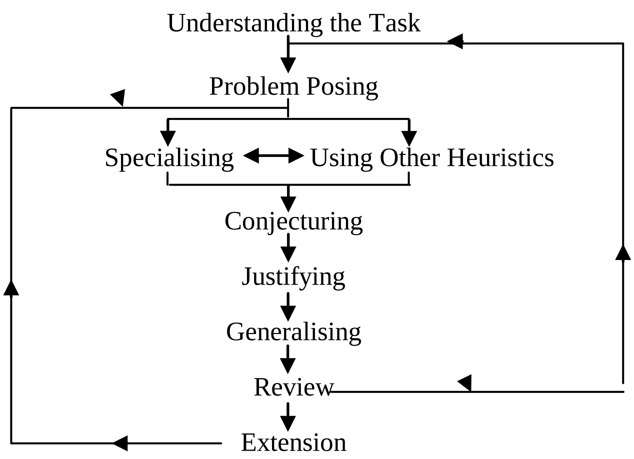

Figure 1. Investigation model of thinking processes

Problem Posing

Silver (1994) observed that “problem posing can occur before, during, or after the solution of a problem” (p. 19). For investigation, problem posing occurs even before the students start to think of a solution because they need to pose their own problems to solve. Brown and Walter (2005) discussed two categories of problem posing. The first category is to accept the given in the task statement. There are at least two types of problems in this category. The first type is to search for patterns, e.g., “Is there a pattern? If yes, what is the pattern?” The second type is to generalise if possible, e.g., “Is there a formula? If yes, what is the formula?” To illustrate the first category of problem posing using Task 2, students can try examples to look for patterns and then generalise without changing the given.

The second category is to extend the task by changing the given using the ‘What-If-Not’ strategy. Kilpatrick (1987) discussed taking something that is a constant in the task statement and let it vary, while Orton and Frobisher (1996) used the “What if” question to extend a problem by changing the given. Using Task 3 as an illustration, students can ask, “*What if* there are four slices to be toasted? *What if* the number of slices the grill can hold is *not* two but three?” This kind of extension can also lead to the finding of a general formula for the time taken to toast the bread. The difference between the process of generalising in the first and second categories of problem posing is that the former does not involve changing the given while the latter requires changing the given to extend the task.

Krutetskii (1976) argued that there were some problems that ‘naturally follow’ from the given task, and he found that high-achieving students were able to pose them directly but low-achieving students were unable to do so. Using Task 3 as an illustration, students may ask, “What is the time taken to toast the three slices?” or even pose the intended problem: “What is the shortest time needed to toast the three slices?” Students are also expected to generalise by asking, “Is there a formula to toast n slices?” In fact, they can even go to the next level of generalisation by asking whether there is a formula to toast n slices if the grill can hold exactly m slices.

Since these problems may be ‘natural’ from the perspective of the person who designs the task but they may not be ‘natural’ for students to pose, there is a need to distinguish the difference. In this paper, this kind of problems will be called ‘logical problems’ that follow from the task because these problems are posed based on a logical sequence. For example, before you can find the least time taken to toast the three slices, you need to find the time taken to toast the three slices using a few methods first. In line with the research question for this paper, an area of interest is whether students find it ‘natural’ to pose such ‘logical problems’ that follow from the given task, including the intended problem.

Methodology

A teaching experiment was conducted with a class of 19 high-achieving students, who had no prior experience in investigation, to develop their processes when they attempted both types of investigative tasks. They were chosen from a bigger pool of high-achieving students from the same school by giving each one of them an investigative task and asking them questions such as whether they had seen this kind of tasks before, and whether they knew how and what to investigate. Only students who answered ‘no’ for all these questions were selected. As it was beyond the scope of the present research to study a large group of students with different academic attainments, high-achieving students were chosen because an initial exploratory study by the author (Yeo, 2008) has suggested that they were more likely to exhibit a wider range of investigation processes to inform the investigation model (shown in Fig. 1) than average or low-achieving students.

However, the same exploratory study has also revealed that most of the high-achieving students with no prior experience in investigation did not really know what and how to investigate because they did not understand the task requirements. Thus the students in the current study were given a two-hour familiarisation lesson before the pretest to teach them *what* to investigate. After the pretest, the students were given five two-hour developing lessons to develop their processes further, i.e., to teach them *how* to investigate. The posttest was conducted within one week after the last lesson. The pretest and the posttest each contained two tasks, one of which was a Type B task. The students were given 30 minutes to do each task. For the pretest, the Type B task was Task 3 described earlier in this paper, while the Type B task for the posttest was as followed:

Task 4: Investigative Task (Sausages)
I need to cut 12 identical sausages so that I can share them equally among 18 people. Investigate.

Unlike traditional textbook exercises where parallel items mean changing the numbers while retaining the same method of solution, parallel items for investigative tasks could not be designed in this way, or else the posttest would just be an exercise for the students with nothing new to investigate. Thus the posttest tasks must have different patterns but still retain the same structure and elicit similar processes as the pretest tasks. For example, Task 3 (Pretest) and Task 4 (Posttest) are of the same Type B, and both involve minimising (find least time and least number of cuts) and generalising using two variables (toast n slices in a grill that can hold exactly m slices, and share n sausages equally among m people).

In addition, each student was videotaped thinking aloud during the tests to capture his or her thinking processes, so as to identify the types of processes in order to study their interaction among one another, and their development. To prepare the students to think aloud during the tests, they practised thinking aloud during the familiarisation lesson and the last developing lesson. The verbal protocols of the students during the tests were then transcribed and coded to identify the types of processes. Four representative transcripts with a total of 1258 codes were given to two experienced coders to code. The inter-coder agreement was 93%, suggesting that the coding scheme developed by the author was reliable enough. The coded transcripts were then manually analysed to study the nature and development of thinking processes in mathematical investigation.

To answer the research question in this study on the kinds of problems posed by the students, there was a need to look beyond the students' test answer scripts to their thinking-aloud protocols, because sometimes the students posed a problem verbally without writing it down in their answer scripts, and it was not easy to guess from their working what their problem was. Occasionally, the students posed a problem verbally but they then changed their minds, so they did not write the problem down in their answer scripts. Therefore, the students' thinking-aloud protocols complemented their answer scripts in providing a fuller picture of the kinds of problem the students posed, so as to answer the current research question more reliably.

Results and Discussion

The logical problems to pose for Task 3 (Pretest), including the intended problem, in sequential order of reasoning as explained earlier in this paper, are:

- Logical Problem 1 (LP1): Find the time taken to toast the three slices of bread.
- Logical Problem 2 (LP2): Find the shortest time to toast the three slices of bread (Intended Problem).
- Logical Extension 1 (LE1): Find the shortest time to toast *n* slices of bread.
- Logical Extension 2 (LE2): Find the shortest time to toast *n* slices of bread if the grill can hold *m* slices.

The logical problems to pose for Task 4 (Posttest) are:

- LP1: Find the number of cuts to share the 12 sausages among the 18 people.
- LP2: Find the least number of cuts to share the 12 sausages among the 18 people (Intended Problem).
- LE1: Find the least number of cuts to share *n* sausages among *m* people.

Unlike Task 3 which has two extensions because it is more logical to first change the number of slices to be toasted than the number of slices the grill can hold, Task 4 has only one extension because it cannot be determined which of the two variables is more logical to change first. In fact, data analysis has shown that most of the students in this study changed both the number of sausages and the number of people at the same time. Another interesting finding for Task 4 was that most students (79%) found it even more natural to pose a problem that was not LP1 or LP2:

- LP0: Find how to cut the 12 sausages to share among 18 people.

This problem is called LP0 since, in sequential order of reasoning, one has to find how to cut the sausages before finding the number of cuts (LP1). Therefore, a similar LP0 was also created for Task 3 as shown in Table 1, which contains a summary of the findings for the present study. Most students posed more than one problem for each task.

Table 1
Summary of Findings (total number of students in this study = 19)

Task 3 (Pretest)	No.	Task 4 (Posttest)	No.
Logical Problem 0 (LP0): Find how to toast the 3 slices	0	LP0: Find how to cut 12 sausages to share among 18 people	15
Logical Problem 1 (LP1): Find time taken to toast the 3 slices	11	LP1: Find number of cuts to share 12 sausages among 18 people	3
Logical Problem 2 (LP2): Find shortest time to toast 3 slices (Intended Problem)	11	LP2: Find least number of cuts to share 12 sausages among 18 people (Intended Problem)	6
Logical Extension 1 (LE1): Find (shortest) time taken to toast *n* slices	7		
Logical Extension 1 (LE2): Find (shortest) time taken to toast *n* slices if grill can hold *m* slices	0	LE1: Find (least) number of cuts to share *n* sausages among *m* people	5
Attempted any kind of extensions	12	Attempted any kind of extensions	19

Problem Posing

For Task 3 (Pretest), data analysis shows that 11 of the 19 students (58%) posed LP1, while 6 students posed LP2, almost immediately after understanding the task. Only two students did not know what to pose. Of the students who posed LP1, 5 of them also posed LP2 almost immediately after solving LP1. Thus a total of 6 + 5 = 11 students (58%) found it natural to pose LP2, which was also the intended problem. For Task 4 (Posttest), data analysis shows that 3 students posed LP0 explicitly, but 12 students just went ahead to find how to cut the sausages without posing LP0 at all. It seems that Task 4 was phrased in such a way that it was very natural for the students to proceed to solve LP0 without even posing it. Thus we might say that a total of 3 + 12 = 15 students (79%) found LP0 natural enough to pose although the majority did not pose it directly. This was unlike Task 3 where no one posed LP0 as the students found it more natural to pose LP1 and LP2 instead.

However, only 3 students (16%) attempted to find the number of cuts (LP1) for Task 4, as compared to Task 3 where it was natural for the majority (58%) to find the total time taken to toast the three slices (LP1), probably because the time for each process, such as putting in or taking out a slice, was given in the task statement. For Task 4, a total of 6 students (32%) posed LP2 naturally: three of them after solving LP1; one of them after solving LP0 without posing LP1; and the last two students after reading the task only once, without even posing LP0 or LP1. For example, one of the last two students said:

> "I need to cut 12 identical sausages so that I can share them equally among 18 people. Investigate. Cut 12, um, maybe, like, the minimum number of cuts." [S17; Posttest]

From the above analysis, the context and phrasing of a task seem to affect whether the students found it natural to pose the logical or intended problems, e.g., fewer students were able to pose LP1 and LP2 for Task 4 as compared to Task 3, despite the fact that Task 4 was a posttest task after 10 hours of lessons to develop students' investigation processes. Moreover, it was not necessary for the students to pose the logical problems in order, e.g., they could pose LP2 directly, without even posing LP0 or LP1.

Extension

Since some students did not find the shortest time (LP2) for Task 3, all students who used any toasting method to find a general formula for the toasting time, not necessarily the shortest time, will be included in the analysis. The findings show that it was only natural for 7 students (37%) to find the time taken to toast *n* slices (LE1). No student attempted LE2, but this might be due to the time constraint of the test: 6 of the 7 students ran out of time when trying LE1 while the last one just managed to find a general formula. Nevertheless, the last student did recognise that his formula is only true for the case where the grill can hold exactly two slices and the timings are as those given in the task statement:

> "And this is the formula for this question. But very sadly, it's only for this one. So, only when you get a question which has [pause] grill it for 30 seconds, take it down will be 5 seconds, and turn it over is 3 seconds, and this must be, and must have, can toast two bread at a time. So then you can get this number and use this formula." [S04; Pretest]

Another 2 students attempted other kinds of generalisation: one of them tried to find a formula to toast *n* slices if it takes *a* seconds to put a slice in or to take a slice out, *b* seconds to toast a slice, and *c* seconds to turn a slice over, for the case where $0 < n \leq 2$ (the latter condition was puzzling); while the other one tried to find the time taken to toast a slice of bread that has *x* number of sides (again it was puzzling how a slice of bread could have, e.g., 5 sides). Nevertheless, a total of 7 + 2 = 9 students (47%) extended Task 3 by generalising. A possible reason why so few students generalised was that most of them still did not fully understand they could change the given in the task statement, even after a two-hour familiarisation lesson. For example, a student asked the invigilator during the pretest:

> "Teacher, the investigation must be, um be, related to the question? Like if they say 3, I investigate finish, can I use 21 slices of bread?" [S12; Pretest]

However, 3 other students extended the task without generalising, e.g., toasting the three slices together in a bigger grill, not a grill that can hold exactly *m* slices. Thus, a total of 9 + 3 = 12 students (63%) extended Task 3 in one way or another.

For Task 4, only 5 students (26%) tried to generalise the least number of cuts (LE1). A possible reason why so few students attempted LE1 was because most of them did not even find the number of cuts for the original task (LP1 or LP2). Nevertheless, another 7 students did try to generalise for other reasons, e.g., to find a general formula for the amount of sausage a person will get, or a general method to cut 6*n* sausages to share equally among 6(*n*+1) people. Therefore, a total of 5 + 7 = 12 students (63%) attempted some kind of generalisation for Task 4, as compared to only 47% for Task 3. However, all the students (100%) extended Task 4 in various ways, as compared to only 63% who extended Task 3. The following show some of the extensions attempted by the students for Task 4:

> "How to share usage of a computer with 9 brothers over a week?" [S02; Posttest]
> "If a sausage is a triangle … if it is a square …" [S06; Posttest]
> "We can use 12 different sizes of sausages." [S18; Posttest]

From the above analysis, the 10 hours of developing lessons had helped the students to be more proficient in extending Task 4 (Posttest) as compared to Task 3 (Pretest), although their attempt to generalise the number of cuts for Task 4 (LE1) was hampered by their inability to pose the logical problem of finding the number of cuts for the original task (LP1 or LP2). Moreover, the quality of the extension for Task 4 was also higher: the students not only changed the numbers in the original task statement of the posttest, but they also changed other variables such as the size and shape of the sausages, and the context to sharing the usage of a computer.

Conclusion

The opening up of mathematical problems to become investigative tasks has provided students the opportunity to pose their own problems to solve. But unlike what Krutetskii (1976) has discovered, some high-achieving students in this study did not find it natural to pose problems that logically follow from the task. The analysis has also revealed that the context of the task and the phrasing of its task statement might affect whether students are able to pose the logical or intended problems, suggesting that teachers could not just design an investigative task by opening up a mathematical problem, as proposed by Frobisher (1994), without considering the possibility of 'losing' the logical or intended problems that they might want their students to solve. Therefore, there is a need for teachers to mull over how to guide their students to pose the intended or logical problems that follow from such investigative tasks, and yet not close up the tasks by restricting the students' freedom and creativity to pose other types of problems to solve. On the other hand, the students had learnt from the teaching experiment how to extend the tasks in different ways, although there was still room for improvement in their ability to generalise. This paper has contributed to the current research through documenting the types of problems that students posed during investigation, and suggesting that these problem-posing processes could be developed in a mathematically enriched environment, although more research needs to be done to study how students cultivate these processes over a longer period of time.

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