Mapping Students' Spoken Conceptions of Equality

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This study expands contemporary theorising about students' conceptions of equality. A nationally representative sample of New Zealand students' were asked to provide a spoken numerical response and an explanation as they solved an arithmetic additive missing number problem. Students' responses were conceptualised as acts of communication and analysed according to their mathematical structure. Specifically, students' spoken explanations were parsed and mapped using the properties of equality. These maps were classified according to their correspondence to the mathematical structure of the given problem. Students gave four different numerical responses and their explanations were interpreted as seven different conceptions of equality. These findings indicate that students' conceptions of equality are more diverse and complex than previous accounts suggest.

Equality is a fundamental concept upon which further mathematics knowledge is built. Mathematics education researchers continue to be challenged in the study and in how they theorise students' understanding of the concept of equality (Dougherty, 2010). Students and teachers speak of equality in words such as *equals* and *is the same as*, these informal phrases connote the state of two quantities being the same. Formally, however, equality is symbolised ideographically as "=" to denote a binary relationship between two arithmetic statements that is reflexive (e.g., 10 = 10 and 10 = 9 + 1 and 10 = 8 + 2 and 10 = 7 + 3), symmetrical (e.g., if 7 + 3 = 10 then 10 = 3 + 7) and transitive (e.g., if 7 + 3 = 10 and 10 = 8 + 2, then 7 + 3 = 8 + 2). While students' errors have been well documented and theorised, students' correct responses lack the same level of detailed scrutiny. Moreover, when a nationally representative sample of Year 8 (12 and 13 year-olds) New Zealand students were asked to solve the additive missing number problem, 7 + 3 = □ + 2, their verbal responses revealed conceptions of equality that appear to be more diverse than previously reported in the literature, and more complex than can be accounted for by dichotomous theorising.

Theoretical Framework

Historically, when students have solved problems involving the equals sign, researchers have theorised students' conceptions of equality dichotomously. Students have been theorised to hold a procedural conception instead of a structural conception of the concept of equality (Brekke, 2001; Falkner, Levi, & Carpenter, 1999; Knuth, Stephens, McNeil, & Alibali, 2006). The additive missing number problem examined in this study, 7 + 3 = □ + 2, is used to illustrate how researchers have reported at least four types of procedural conceptions of the equals sign that result in incorrect numerical responses. One student conception of the equals sign is as a signal to perform an action (Behr, 1976). In this case, a student will add the two numbers that precede the equals sign, therefore 7 + 3 = □ + 2 is spoken as "7 + 3 = 10" so the missing number is 10. A second conception is a prompt to execute a procedure adding up all numbers in the problem and placing an answer after the equals sign, therefore 7 + 3 = □ + 2 is spoken as "7 + 3 + 2 = 12", so the missing number is 12 (Kieran, 1980). A third conception is as an operator-separator, where an arithmetic statement is not complete without a solution to the right of an equals sign (Baroody & Ginsburg, 1983). In this case, the equals sign is viewed as a placeholder,

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therefore 7 + 3 = □ + 2 is spoken as "7 + 3 = 10 + 2 = 12", so the missing number is 12. A fourth conception is when the equals sign forms part of a restricted notational structure (Seo & Ginsberg, 2003). In this case, "+ 2" is said to be extraneous, therefore 7 + 3 = □ + 2 is spoken as "7 + 3 = 10", so the missing number is 10. These procedural types of conceptions have been reported as teachers' conceptions of equality as well (Attorps, 2006). While procedural conceptions of equality have been well documented and theorised in the literature, structural conceptions have not. When a student gives the correct missing number, 8, and says "7 + 3 = 8 + 2", the features of that student's structural conceptions of equality remain unclear.

Researchers have expanded upon the dichotomous foundation by inquiring about students' conceptions of equality as legitimate attempts to participate in mathematical activity (Dayvdov, 1990; Roth, 2012; Sfard, 1998). These communicative acts are viewed as additional sources of information about students' conceptions of equality that can be examined in addition to correct and incorrect numerical responses or calculation sequences. To analyse these communicative acts, researchers are using mathematical structure as an analytic tool (Caspi & Sfard, 2012). Mathematical structure is "the identification of general properties which are instantiated in particular situations as relationships between elements" (Mason, Stephens, & Watson, 2009, p. 10). By including a focus on mathematical structure, the formal properties of equality become foregrounded as the object of study. Studies conducted from a participative and mathematical structure perspective draw attention to the fact that the reflexive and symmetric properties of equality have often been neglected in arithmetic teaching contexts (Attorps & Tossavainen, 2007). Likewise, student's conceptions of equality have been documented to lack transitive (Godfrey & Thomas, 2004; Jones, 2009) and symmetrical properties (Jones, Inglis, Gilmore, & Dowens, 2012). Researchers have also enlarged the scope of possible inquiry by conducting longitudinal studies with cohorts of students that examine the transition in students' conceptions of equality as they solve problems in increasingly algebraic-like contexts (Stephens & Xu, 2009; Xu, Stephens, & Zhang, 2012). Despite these efforts, however, there is a paucity of literature that theorises how students develop formally recognisable mathematical concepts (Caspi & Sfard, 2012). This study contributes empirical evidence and builds upon contemporary theorising by using mathematical structure, and specifically, the formal properties of equality, as an analytic tool.

Method

Participants and Instrument

The Educational Assessment Research Unit at the University of Otago collected the mathematics assessment data used in this study as part of a project that was conducted from 1995 to 2010. The National Educational Monitoring Project assessed students in Year 4 (8 and 9 year-olds) and Year 8 (12 and13 year-olds) and reported what they knew and could do at those two levels of schooling. In 2009, a nationally representative sample (N =422) of Year 8 students were asked to respond to 56 number knowledge and algebra tasks (Crooks, Smith, & Flockton, 2010). One of those tasks, *Link task 2*, consisted of three additive arithmetic missing number problems, shown in Figure 1. *Link task 2* was video recorded while students solved those problems in a one-to-one interview context with an adult assessor. Specifically, students were shown the three problems in Figure 1 and prompted by the assessor to give spoken responses and explanations as they solved those problems. The assessor's protocol is shown in Figure 2.

5 = 3 + □

7 + 3 = □+ 2

2 + □ = □+ 6

Figure 1. The three arithmetic missing number problems shown to students in *link task 2.*

For *link task 2*, the object of the assessment was for students to engage in mathematical problem solving, and more specifically, the outcome was for students to offer a verbal solution and explanation for each problem. A subsample of student data (N = 122) was randomly sampled from the original data set. Only the spoken responses to the second problem in the task (i.e. 7 + 3 = □ + 2) provide the source of data analysed in this paper.

1. *I'm going to show you some problems with numbers that are missing.* (Hand student card and point to problem 1. 5 = 3 + □)
2. *Could the missing number be 8?* (Record student's answer.)
3. *Explain why you say that.*
4. (Point to problem 2. 7 + 3 = □+ 2) *What is the missing number?* (Record student's answer.)
5. *Explain why you say that.*
6. (Point to problem 3. 2 + □ = □+ 6) *What do you think the two missing numbers could be?* (Record student's answer.)
7. *Could you have any other numbers?* (If student answers yes ask "*what numbers?*")
8. *Explain why you say that.* (Record student's answer.)

Figure 2. Assessor protocol for *link task 2* with verbal prompts in italics and non-verbal prompts in parentheses.

Analysis

Student's spoken responses were transcribed verbatim from digital copies of the original video recordings and were entered manually into an Excel spread sheet.

Numerical responses. Transcribed responses to the prompt, "What is the missing number?" were interpreted as solutions to the missing number problem and recorded as numbers. If the transcribed response contained was more than one solution, then those missing numbers were noted in the spread sheet. The number and frequency of the missing numbers spoken by students were calculated.

Parsing explanations. Transcribed explanations to the prompt, "Explain why you say that" were analysed using parsing trees because accurate representation of students' explanations in mathematical notion was not possible in all cases. Students often repeated operations, changed their missing number, or re-phrased their explanations, therefore another mode of representation was chosen. Traditionally, parsing is a technique used by linguists to analyse text. If a sentence is parsed, then each word is identified and classified by its form, function, and syntax in that particular sentence. Computational linguists developed parsing trees to visualise how a particular sentence could be produced from its given syntax (Grune & Jacobs, 2008). Because parsing trees allow the syntactic and semantic features of sentences to be identified, classified, and visualised, this technique was adapted and applied to the transcribed explanations of the mathematical sentence, 7 +

3 = □ + 2. Instead of the term “parsing trees” the mathematical illustrations constructed are referred to as parsing maps, to avoid confusion with their computational linguistic origins.

Parsing procedure. Transcribed explanations were parsed into three components: mathematical form, sematic function, and syntactical features. The example of a student response, “seven plus three equals ten” is used to illustrate this process. Mathematical forms identified were numbers, operations, and relationships. Mathematical forms were recorded with mathematical symbols so that “seven”, “three”, “ten” were identified as “7”, “3”, and “10”, “plus” was identified as “+”, and “equals” was identified as “=”. Semantic function was identified according to how each word or combination of words could be classified as parts of speech, phrases, or clauses. Parts of speech were used to classify each word as a verb, noun, conjunction, preposition, adverb, adjective, pronoun, or other. If the part of speech identification was ambiguous, such as for the word “equals”, then more than one designation was given. Thus the elements in the explanation “seven plus three equals ten” was correspondingly classified as noun, preposition, noun, verb/noun, noun. When several words functioned together as a semantic unit with either a noun or a verb, they were classified as types of phrases. The words, “seven plus three” were classified as an addition phrase. When several words functioned together as a semantic unit with a noun and a verb, they were classified as types of clauses. The words, “equals ten” were correspondingly classified as an equality clause. Sematic functions were recorded as parts of speech nested in sets of types of phrases and clauses. Syntactical features were identified comparing the mathematical form and sematic function of a particular explanation to those associated with the formal mathematical structure of the given problem. So “seven plus three equals ten” with one addition phrase and one equality clause was compared to “seven plus three equals eight plus two” with two addition phrases and one equality clause. Syntactical features were recorded as the similarities and differences between the formal mathematical structure of the given problem and the mathematical structure of a student’s explanation. A similar syntactical feature for “seven plus three equals ten” was the addition phrase, “seven plus three”. The differences were the type of equality clause, “equals ten” as opposed to “equals eight plus two”, and a numerical value, “10” as opposed to and addition phrase, “eight plus two”. Using this procedure, each student’s explanation was classified along the dimensions of mathematical form, semantic function, and syntactical features.

Mapping procedure. To facilitate the interpretation of the mathematical structure of each explanation, the parsing classifications were used to construct parsing maps that illustrated the properties of equality of each student’s explanation. For example, the mathematical structure in the statement, 7 + 3 = □ + 2, involves formal properties of equality, operations, and number. These formal properties formed an initial set of guidelines used to construct the parsing map shown in Figure 3. First, the parsing components corresponding to the formal properties of equality were identified and used to form the starting point of the map. In this case, there are two statements that are quantitatively equal, 7 + 3 and □ + 2, therefore the equality symbol became the first node and was written with two branches descending from it. Secondly, the parsing components corresponding to the formal properties of operations were identified and used to form the first level. In this case, both statements involve addition and each operation has two addends, 7 + 3 and □ + 2, therefore each addition symbol became a node and was written with two branches descending from it. Thirdly, the components relating to the formal properties of numbers were identified and used form the nodes at the second level. In this case, the numbers are shown with no further downward branches because the formal properties of numbers are "hidden" within the numerals used. In this way the mathematical structure for the missing number problem, $7 + 3 = \square + 2$ is accounted for as hierarchically nested sets of illustrated properties. The initial set of guidelines was adapted to include all of the various ways students expressed their explanations. Where students did not explicitly articulate a relationship between parsing components, they were drawn as dashed lines.

$$7 + 3 = \square + 2$$

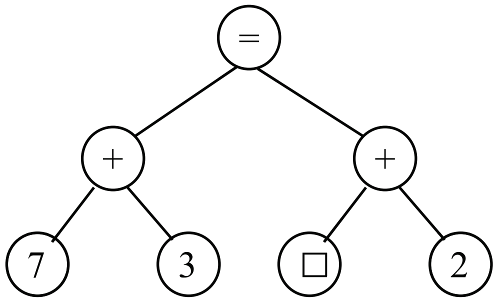


Figure 3. Parsing map for problem 2 in *link task 2*.

Results

Of the 122 spoken responses to the question, "What is the missing number?" Year 8 students' speech was found to contain only four different numerical responses; 1, 8, 10 and 12. The correct missing number, 8, was identified in 54 (44%) of the responses. The other 68 (56%) missing numbers spoken were incorrect. These findings are summarised in Table 1. While the responses, 8, 10, and 12 appear accounted for by the dichotomous framework, the response of 1 is not. A conception that is a variation of restricted notational structure (Seo & Ginsberg, 2003), may account for a response of 1 and could be the mathematical structure that is associated the teaching and learning of basic addition facts (i.e., 1 + 2 = 3). Another feature of the results not accounted for by the dichotomous framework are the multiple solutions offered by 11 (9%) of students.

Table 1
Frequency and percent of each missing number spoken by Year 8 students (N = 122) to the prompt, "What's the missing number?" and shown 7 + 3 = □ + 2

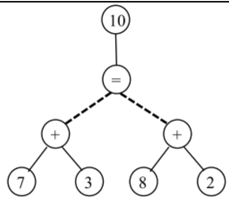
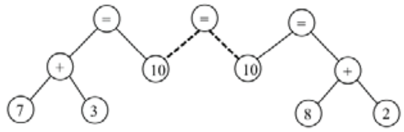
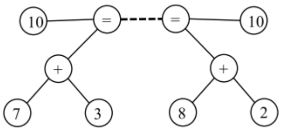
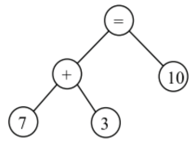
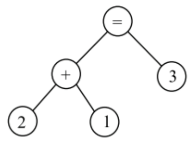
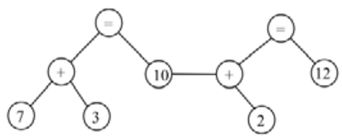
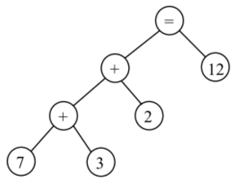
Response	Frequency	Percent
1	3	2%
8 (correct)	54	44%
10*	43	35%
10 or 1	1	1%
10 or 12	10	8%
12	11	9%

* One student responded that 9 was the missing number however the explanation revealed that the student had made a computation or recall of basic facts error.

Of the 122 spoken explanations to the prompt, “Explain why you said that”, 7 types of conceptions of equality were identified. Representative samples of each of the 7 types are shown in Table 2.

Table 2

Representative samples of Year 8 student’ spoken responses to the prompt, “Explain why you say that”, presented by transcribed response, frequency (percent), mathematical equivalent, parsing map , and equality type for link task 2, problem 7 + 3 = □ + 2.

Transcribed Response	Frequency (Percent)	Mathematical Equivalent	Parsing map	Equality Type
Ten, um, ten are the same thing, seven plus three, eight plus two.	9 (7%)	10 = 10, 7 + 3, 8 + 2		Proto-equivalence
Seven plus three is ten and then it would be eight plus two is ten as well.	19 (16%)	(7 + 3 = 10) = (8 + 2 = 10)		Explicit quantitative
Seven plus three is ten and eight plus two is ten.	25 (20%)	7 + 3 = 10 (=) 8 + 2 = 10		Implicit quantitative
Seven plus three equals ten.	36 (30%)	7 + 3 = 10		Restricted action
Two plus one equals three.	3 (2%)	2 + 1 = 3		Known fact
Seven plus three equals ten obviously they are trying to get to twelve, need ten in the middle to get to twelve.	26 (21%)	7 + 3 = 10 +2 = 12		Operator-separator
I added seven plus three plus two and came up with answer twelve.	4 (3%)	7 + 3 + 2 = 12		Procedural

As expected, all of the explanations associated with incorrect solutions offered by students can be accounted for by previous theorising but with one profound difference; all are viewed as equality types rather than misconceptions because of the participative approach framing this inquiry. The equality type descriptors acknowledge the history of that particular conception type and emphasise its principle mathematical feature: procedural (Kieran, 1980), operator-separator (Baroody & Ginsburg, 1983), restricted action (Behr, 1976; Kieran, 1980), or known fact (Seo & Ginsburg, 2003). Explanations associated with the correct solution offered by students are accounted for by previous theorising, in a general sense only, because they are structural conceptions. The dichotomous framework, however, does not account for range of correct types documented in this study. The equality type descriptors emphasise the principle mathematical feature of that particular conception type: proto-equivalence, explicit quantitative equality, or implicit equality.

Discussion

Results indicate that students' conceptions of equality are more diverse and complex than previous accounts suggest. The parsing maps are a novel analytic tool that may be useful for examining other mathematical ideas and illustrating the features of specific problems with students in classrooms. When students' explanations were examined using a participative approach (Dayvdov, 1990; Roth, 2012; Sfard, 1998), a greater diversity and complexity of conceptions of equality became discernible. If only the types of numerical responses had been examined, it would have appeared that students were demonstrating a structural conception when they gave a correct solution (Brekke, 2001; Falkner, Levi, & Carpenter, 1999; Knuth, Stephens, McNeil, & Alibali, 2006) and were demonstrating one of a least four procedural conceptions of equality when they gave an incorrect response (Baroody & Ginsburg, 1983; Behr, 1976; Kieran, 1980; Seo & Ginsburg, 2003). In this study, it appears that about one in ten students may be appreciating different features of the mathematical structure in the problem and they may be confused by the multiple conceptions of equality that are possible for those features. Students' ambiguous responses have been a feature of inquiry (Caspi & Sfard, 2012) and could be the focus of future inquiry with the type data used in this study. This study also drew upon contemporary studies that have examined the properties of equality using a participative approach (Attorps & Tossavainen, 2007; Caspi & Sfard, 2012; Godfrey & Thomas, 2004; Jones, 2009; Jones, Inglis, Gilmore, & Dowens, 2012; Stephens & Xu, 2009; Xu, Stephens, & Zhang, 2012) and theorised students' conceptions using mathematical structure (Mason, Stephens, & Watson, 2009). Parsing maps were an analytic tool used to discern at least three variations of student's structural conceptions of equality and helped to reframe procedural conceptions as informal types of conceptions rather than misconceptions. It appears that students are using at least three types of conceptions of equality to solve missing number problems successfully. Not all structural conceptions of equality appear to be equal.

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