Translating Between and Within Representations: Mathematics as Lived Experiences and Interactions

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Students develop understanding of mathematics when they translate between and within different mathematical representations. This paper explores a student-generated story and content descriptors from the Australian Curriculum: Mathematics to highlight how primary school students can represent mathematical concepts through exploring the links between everyday physical objects, pictures, oral/written language, models and mathematical symbols. This active experience enhances the students' capacity to represent mathematical concepts and ideas, symbolise these, and eventually learn to abstract and generalise.

The Australian Curriculum: Mathematics (ACARA, 2011), highlights that mathematics is composed of multiple but interrelated and interdependent concepts which students apply beyond the mathematics classroom. Ellis and Berry III (2005) argue this new perspective regards mathematics as a set of logically organised and interconnected concepts that come out of human experience, thought and interaction. These mathematical concepts can be accessible to all students if learnt in a cognitively connected and culturally relevant way. The emphasis focuses on mathematics as a part of human experience and interaction, where students understand mathematics when they have the opportunities to: share common experiences with and around mathematics that allow them to meaningfully communicate about and form connections between important mathematical concepts, and engage in critical thinking about the ways in which mathematics can be used to understand relevant aspects of their everyday lives.

According to Van de Walle and colleagues (2013) understanding mathematics is a measure of the quality and quantity of connections that a new mathematical idea has with existing ideas. The greater the number of connections a student makes to a network of ideas, the better the understanding. This means that the more opportunities students are given to think about, develop and test an emerging idea, the better chance they will have to correctly form and integrate it into a rich web of concepts and therefore develop deeper understanding. Understanding mathematics involves building robust knowledge of adaptable and transferable mathematical concepts, the making of connections between related concepts, the confidence to use the familiar to develop new ideas, and the why as well as the how of mathematics (National Curriculum Board, 2009).

Mathematical representations as human experience and interaction

The development of representation and systems of representation is central to teaching and learning of mathematics. Goldin (2002) highlights a representation is a configuration that can represent something else in some manner, for example, a word can represent a real-life object, a numeral can represent the cardinality of a set, or the same numeral can represent a position on a number line. The representing configuration might, for instance, act in place of, be interpreted as, connect to, correspond to, denote, depict, embody, encode, evoke, label, link with, mean, refer to, resemble, serve as a metaphor for, signify, stand for, substitute for, or symbolise the represented one. Goldin suggests the process of abstraction is one that involves reaching the autonomous stage in the functioning of a

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representational system. The representational perspective permits us to relinquish the idea that mathematics in context is somehow the opposite of ‘formal abstract’ mathematics.

Lesh and colleagues (2003) outline five different ways to represent mathematical ideas: real-world situations, manipulative models, pictures, oral/written language, and written symbols. These mathematical representations play an important role in mathematical activity. They are increasingly seen as a useful tool for building and communicating both information and understanding. Siemon and colleagues (2011) classify these representations as either internal modes (in our mind) or external modes (in the world around us). Strengthening the ability for students to move between and among these representations can improve their understanding and retention of mathematical concepts. Lesh and colleagues suggest students who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computation. Developing students’ capacity to move between and within different mathematical representations can be a focus for contemporary mathematics classes.

Contemporary mathematics classes can be physical, blended or virtual learning spaces. The physical learning space is a face-to-face meeting with no use of emerging technology and/or the internet for teaching and learning. Virtual learning environment is fully online learning and enables students to log into synchronous or asynchronous learning opportunities where there is an internet connection. Blended learning environment involves combining a physical and virtual environment, where classroom face-to-face time is reduced and some time devoted to online learning (Bates & Poole, 2003). There is need to develop students’ capacity to move between and within a variety of mathematical representations in these different learning spaces. A constant exposure to digital technologies, gadgets, games, and mobile devices has arguably evolved a new breed of student, the ‘natives’ who think and process information fundamentally differently from their predecessors, the ‘immigrants’, whose interaction with these tools is not innate (Cobcroft, et al., 2006).

In contemporary mathematics classes, there is need to develop “understanding and thoughtful action that deep mathematical learning requires” (National Numeracy Review Report, COAG, 2008, p. x ii). Skemp (1978) highlights two distinct meanings of understanding some aspect of mathematics: instrumental and relational understanding. Instrumental understanding equates with simply knowing a fact or being able to perform a mathematical procedure. Kilpatrick and colleagues (2001) named the relational understanding strand conceptual understanding. Conceptual understanding involves knowing what, how and why a procedure works or a fact is true. Conceptual understanding is never complete because there are always new connections or relations that can be made. Conceptual understanding provides a basis on which further understanding can be built by extending and adding to existing connections or relations. According to Reys and colleagues (2004) a curriculum that emphasises many physical models and representations: pictorial, manipulative, verbal, real-world, and symbolic is more successful in aiding students’ development of conceptual understanding.

Contemporary learning in mathematics needs to target for conceptual understanding by building on m eaningful ideas and multiple representations on r ich and challenging tasks supported by collaborative discussion and personal success (Siemon et al., 2011). According to Skemp (1978) a concept is a way of processing data which enables the user to bring past experience usefully to bear on the situation. Thus conceptual understanding is indicated by the quality and quantity of connections that are made between new and existing ideas. Teachers need to realise that rushing to apparent proficiency of procedural skills (rules of mathematics) at the expense of sound conceptual development needed for sustained and ongoing mathematical proficiency is not very helpful to students. While there is a place for procedural learning, for example, the need to recognise the number names and symbols and perform procedures; most learning in mathematics should target for conceptual understanding. Bobis and colleagues (2013) observe mathematics is about seeking patterns and relationships, representing and symbolising these ideas, and eventually learning to abstract and generalise.

According to Demetriou and colleagues (2011) as children grow older, they become able to deal with increasingly more complex representations. The development is a continuous process from emergence to differentiation and integration of new representations. The emergence of language during the second year of life brings representations into focus so they can be reflected upon and elaborated. At about the age of three to four years, children start to differentiate representations from each other and from the objects they represent (DeLoache, 2000). These pre-schoolers start to differentiate objects from their representations and develop ideas in the various environment oriented domains. Another important milestone comes at about the age of seven to eight years. With practice and increasing awareness about representations, children at this milestone begin to realize that representations constrain each other (Demetriou & Kazi, 2006).

A third milestone comes at the beginning of adolescence, when adolescents become able to construct conditional representations (Demetriou et al., 2011). Conditional representations enable the students to view systems of representations from the point of view of each other. They can integrate at least two dimensions; for example, they can grasp proportionality and formulate alternative hypotheses which they can test. This increasing complexity necessitates the elaboration of relations between and within representations, which results into abstractions, which are subsequently encoded and processed as representations, thereby leading to increasingly abstract systems of representations.

The Australian Curriculum: Mathematics provides advice across four year groupings on the nature of learners and the relevant curriculum: foundation to year 2; years 3 to 6; years 7 to 10; and senior secondary. Foundation to year 2 lays the foundation for learning mathematics which is vital to future progression. Students at this level can access powerful mathematical ideas relevant to their everyday lives and also learn the language of mathematics. Years 3 to 6 emphasise the importance of students developing meaningful and purposeful mathematical experiences relevant to their lives. Students at this stage still require active experiences that allow them to construct key mathematical ideas, but also gradually move to using models, pictures and symbols to represent these ideas. Years 7 to 10 mark a shift in mathematics learning to increasingly abstract systems of representations. Through activities such as the exploration, recognition and application of patterns, the capacity for abstract thought can be developed and ways of thinking associated with abstract ideas can be illustrated.

Excerpt of a student-generated example: the story of the seven puppies

An excerpt adopted from Chigeza (2012) is used to highlight mathematical concepts through exploring the links between everyday physical objects (the seven puppies), oral/written language, pictures, models and mathematical symbols. Perger (2011) uses stories in mathematics programs with year 7 students and observes that their learning of mathematical concepts is enhanced when the students are given the opportunity to identify and talk about the mathematics in the story. Perger concludes that the use of stories can be a powerful tool in both motivating and consolidating mathematical knowledge. The excerpt adopted from Chigeza concerns four middle school students: Liz, Pen, Tim and Ron discussing and exploring the idea of unknowns and variables in equations.

The excerpt:

Liz: An unknown is defined as the value which is to be discovered by solving an equation. But we need to distinguish between an unknown and a variable.

Pen: We can use letters from the alphabet to represent both an unknown and a variable. The difference is that an unknown has a specific value one can calculate or figure out, but a variable represents a range of numbers.

Ron: I am still confused. Can you give me an example?

Tim: When I visited you at home last weekend Ron and you showed me your seven puppy dogs. The first time I looked at them, four were feeding and the rest were playing.

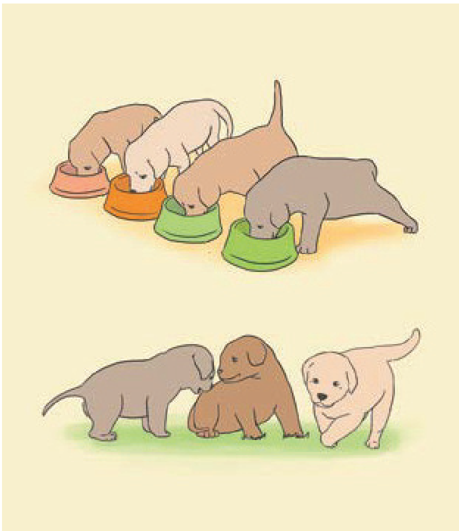

Figure 1. The seven puppies (Chigeza, 2012).

Liz: We can use a number sentence: __ + 4 = 7 to represent what Tim has just described.

Pen: Instead of putting the __ in the number sentence we can decide to use any letter from the alphabet. If we decide to use x, then we can represent it as an equation: $x + 4 = 7$ and we say x is standing in for an unknown which we can figure out.

Tim: That is, if x represents the number of puppies playing, we can represent the number of puppies playing and eating in the equation: Find the unknown: x, the number of puppies playing in the equation: $x + 4 = 7$.

Pen: Now if we say x represents the number of puppies playing and y represents the number of puppies feeding. Then we have the equation $x + y = 7$. If we were to check on the puppies every minute or two, of course knowing puppies, some feeding will go on to play and those playing will go to feed. Now, x and y are variables.

Ron: This is all starting to make some sense now.

Tim: Our everyday world we explored in junior school has not really changed now that we are in middle school. What has changed is how we attempt to represent it.

Making connections within and between mathematical representations

It is important for students in junior and middle school to explore and represent number and algebraic ideas using physical objects, models and symbols. Students should engage in critical thinking about the ways in which mathematics is used to understand relevant aspects of their everyday lives (ACARA, 2011; Ellis & Berry III, 2005). To explore this notion, the content descriptors from the Australian Curriculum: Mathematics are used to highlight how students in years 2, 4 a nd 7 can use the story of the seven puppies to meaningfully communicate mathematical ideas and translate within and between the different and increasingly complex mathematical representations.

In year 2, students are encouraged to solve problems by using number sentences for addition or subtraction (ACARA, 2011). At this level, students can access powerful mathematical ideas relevant to their current lives and also learn the language of mathematics vital to future progression. To make connections between existing and new representations and develop conceptual understanding (Kilpatrick et al., 2001), students can starts from real-world scenario to pictorial to verbal and then develop symbolic representation of the mathematical concepts and ideas (Reys et al., 2004). For example, students used a number sentence: $\_\_ + 4 = 7$ to describe and represent what the seven puppies are doing (Chigeza, 2012). Students at this level can explore as individuals or in small groups similar introductory stories and activities to connect with prior knowledge and clarify their mathematical representations. According to Lesh and colleagues (2003) students who successfully move between and among these representations improve their understanding and retention.

In year 4, students are encouraged to use equivalent number sentences involving addition and subtraction to find unknown quantities (ACARA, 2011). At this level, students still require active experience, but there is a trend to move to using models, pictures and symbols to represent mathematical concepts and ideas. Using the story of the seven puppies, students at this level can discuss the use of x to represent the puppies that are playing in the equation: $x + 4 = 7$. The students can also discuss and ascertain that the equation represents the scenario of the puppies, and x can also be called an unknown which they can figure out. As observed by Bobis and colleagues (2013), mathematics involves representing, symbolising, abstracting and generalising ideas. This movement to integration of new and increasingly complex representations is a continuous process (Demetriou et al., 2011) and the more connections students make between existing and new mathematical representations, the more they enhance their understanding of the mathematical concepts and ideas (Van de Walle et al., 2013).

In year 7, students are encouraged to: solve simple linear equations, and introduce the concept of variables as a way of representing numbers using letters (ACARA, 2011). At this level, students can focus on d eveloping more abstract representations, and can integrate at least two dimensions of representations. Demetriou and colleagues (2011) observe that this increased representational skill enable the students to recognise different patterns and why these patterns apply in the situations. Using the story of the seven puppies, students can use x to represent the number of puppies playing and y to represent the number of puppies feeding. They can also realise that x and y are variables representing the changing scenario of puppies that are playing and feeding as represented in the equation $x + y = 7$. Students at this level need to build on t heir prior representation to develop and integrate new and increasingly complex representations. According to Goldin (2002) developing mathematical representations in context enables students to understand abstract mathematical concepts. Rushing the students to perform mathematical procedures (i.e. following the rules of mathematics) at the expense of developing their increasingly complex systems of mathematical representations, and sound conceptual development is not very helpful for them (Skemp, 1978).

Conclusion

This paper has highlighted that students develop understanding of mathematics when they translate between and within different mathematical representations, and that developing this representational capacity permits us to relinquish the idea that mathematics in context is somehow the opposite of 'formal abstract' mathematics. The quality and quantity of connections that students make between new mathematical ideas with existing ideas is a measure of their understanding. In contemporary mathematics classes, which can be physical, blended or virtual learning spaces, there is need to develop students' capacity to move between and within the mathematical representations in these different learning spaces. The paper employed content descriptors from the Australian Curriculum: Mathematics to highlight how students in years 2, 4 and 7 can use the story of the seven puppies to meaningfully communicate mathematical concepts and translate within and between the different and increasingly complex systems of mathematical representations.

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