Making Connections Between Multiplication and Division

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This paper reports on 13 Grade 3 students' approaches to partitive and quotitive division word problems. Of particular interest was the extent to which students drew on their knowledge of multiplication to solve division problems. The findings suggest that developing a relationship between multiplication and division is more significant than differentiating the types of division.

Teachers in the early and middle primary grades spend much time exploring partitive and quotitive division with their students. Some teachers focus more on partitive division than quotitive division in the belief that it reflects young children's everyday experience of division. In this paper I argue that students need to understand the concept of division and its inverse relationship to multiplication rather than whether a task involves partitive or quotitive division.

Theoretical Background

The notion that division of whole numbers can be interpreted in two different ways reflects its relationship to multiplication, namely division by the multiplier (partitive model) and division by the multiplicand (quotitive model). In the partitive model, commonly referred to as the sharing aspect, the number of parts or subsets is known whereas the size of the subsets is unknown (Fischbein, Deri, Nello & Merino, 1985; Greer, 1992). Using this partitive model, 12 ÷ 4 can be represented as a word problem 'Twelve lollies are shared equally among 4 children. How many did they each receive?' In the quotitive model, otherwise referred to as measurement division, the size of the subsets or parts is known whereas the number of subsets is unknown (Fischbein et al., 1985; Greer, 1992). Interpreting 12 ÷ 4 in this way using the same context as previously would be 'There are 12 lollies and each child receives four. How many children will receive lollies?' In this situation, the dividend and divisor are lollies, which can be 'measured' or partitioned into groups of 4. These examples illustrate the different actions involved, in the partitive model the action is sharing into 4 equal groups, whereas in the quotitive model the action is partitioning the whole into equal groups of 4. As noted by Squire and Bryant (2002) the role of the size of the portion and the number of recipients actually reverses in partitive and quotitive division.

To understand division requires more than knowledge of sharing out a collection equally: it requires an understanding of the relationship between the dividend, divisor and the quotient, and the role of each in a division problem (Correa, Nunes, & Bryant, 1998). Squire and Bryant (2002) argue that students need experiences with a range of problem types to develop their conceptual understanding of the multiplicative relationships inherent in a problem. Neuman (1999) suggests that students might have less difficulty in the middle primary years if division was introduced before or parallel to, multiplication.

A consistent theme in the literature relating to children's solution strategies is that young children draw on their intuitive strategies to solve division word problems. These intuitive strategies include direct modelling and unitary counting, repeated addition or subtraction, and multiplication or division facts (e.g., Mulligan & Mitchelmore, 1997). These studies found that students use different strategies for each form of division whereas

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Brown (1992) reported that Grade 2 children tend to solve partitive problems using grouping rather than sharing strategies. However, Murray, Oliver and Human (1992) found the children's solution strategies (Grades 1 to 3) initially model the problem structure but with experience they are more flexible in the strategies they use and ignore whether it is a partitive or quotitive problem. Studies of students in Grades 4 to 6, involving numbers beyond the multiplication fact range (e.g., Heirdsfield Cooper, Mulligan, & Irons, 1999) found that students' intuitively use multiplication for solving both partitive and quotitive division problems.

Aspects of the study reported in this paper have been reported earlier (Downton, 2009). The intention of this paper is to investigate the question: What strategies do students in Grade 3 use to independently solve quotitive and partitive division word problems? The excerpts of students' solution strategies indicate a natural tendency to use multiplication.

Method

This paper draws on one of the findings of a larger study of young children's development of multiplicative thinking. The study involved Grade 3 students (aged eight and nine years) in a primary school located in suburban Melbourne. In terms of mathematical achievement, the spread of students overall was similar to statewide results for the systemic assessment. All the students in the grade were interviewed prior to the commencement of the study using the counting, addition & subtraction strategies, and multiplication & division strategies domains of the Early Numeracy Interview (Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough, Montgomery, Roche, Sullivan, Clarke, & Rowley, 2002). The resultant "growth point" data were used to identify 13 students in the class, four at either end of the scale and five in the middle, to participate in the study. This number of students provided a representative sample of the grade in terms of initial achievement. The author interviewed the 13 students using a one-to-one, task-based interview to gain insights into their understanding of and approaches to multiplicative problems. The findings of a subset of these results are reported in this paper.

Instruments

The author developed a division task-based interview which consisted of eight tasks in the form of word problems. Two problems across the four semantic structures identified by Anghileri (1989) and Greer (1992) were used: Equal Groups, Allocation/Rate, Rectangular Arrays, and Times-as-Many. Each category included both a partitive and quotitive problem to identify whether there was a relationship between the strategies students chose, the semantic structures, and the division type. In addition, each task consisted of three levels of difficulty, rated as easy, medium or challenge, established from pilot testing. The number combinations (otherwise referred to as number triples) were deliberately chosen to gauge whether there was any relationship between the strategies chosen and the numbers within a task. The duplication of some number triples, both within and across levels of difficulty, was designed to gauge whether students used the same strategies across different semantic structures, and /or forms of division. The number triples (number combinations for dividend, divisor, quotient) were deliberately chosen with some repetition both within and across levels of difficulty. For example, the number triple 24, 6, 4 was in a Rectangular Array quotitive task at the easy level of difficulty and in an Equal Groups quotitive task at the medium level.

As indicated in the literature, the Rectangular Array semantic structure does not have two distinct forms of division problems (Greer, 1992). However, for the purposes of this study, two similar tasks under the guise of partitive and quotitive division were included. This was to identify whether students used the same strategy across both tasks. Table 1 lists the challenge questions chosen for the division interview, noting the type of division (partitive or quotitive). The numbers (dividend and divisor) used in the medium and easy questions are in brackets.

Table 1
Division Interview Whole Number Word Problems

Semantic structure	Aspect of division	Problem
Equal Groups	Partition	I have 48 cherries to share equally onto 3 plates. How many cherries will I put on each plate? (M 18, 3; E 12, 3)
	Quotition	72 children compete in a sports carnival. Four children are in each event. How many events are there? (M 24, 4; E 12, 4)
Allocation/ Rate	Partition	I rode 63 kilometres in 7 hours. If I rode at the same speed the whole way, how far did I ride in one hour? (M 28, 7; E 15, 5)
	Quotition	I have 90 cents to spend on stickers. If one packet of stickers cost 15 cents how many packets of stickers can I buy? (M 60, 5; E 30, 5)
Rectangular Arrays	Partition	One hundred and two pears are packed into the fruit box in 6 equal rows. How many pears are in each row? (M 54, 6; E 24, 4)
	Quotition	I cooked 84 muffins in a giant muffin tray. I put 6 muffins in each row, of the tray. How many rows of muffins on the tray? (M 36, 4; 4; E 20, 4)
Times-as-Many	Partition	Sam read 72 books during the readathon, which was 4 times as many as Jack. How many books did Jack read? (M 36, 4;E20, 4)
	Quotition	The Phoenix scored 48 goals in a netball match. The Kestrels scored 16 goals. How many times as many goals did the Phoenix score? (M 28,7; E 18, 6)

Interview Approach

Each interview was audio taped and took approximately 30 to 45 minutes, depending on the complexity of students' explanations. The problems were presented orally and students were encouraged to work out the answers mentally. However, paper and pencils were available for students to use at any time. Generous wait time was allowed and the researcher asked the students to explain their thinking and whether they thought they could work the problem out a quicker way. Once a response was given the student was asked to explain his/her thinking and record a number sentence on paper. Responses were recorded and any written responses retained. Students had the option of choosing the level of difficulty to allow them to have some control and to feel at ease during the interview. If a student chose a challenge problem and found it too difficult, there was an option to choose an easier problem.

Analysis

Providing students with a choice of tasks contributed to the richness of the findings and also added to the level of complexity both in the analysis and presentation of data. The researcher was interested in knowing both the approaches students used and components of the task that influenced their strategy choice. An extensive analysis was undertaken of each of the task components (e.g., semantic structure, level of difficulty, number triples). Due to the limited space the results of the analysis presented in this paper pertain only to the challenge level of difficulty.

The data were coded for two purposes, first to ascertain student performance and second to identify student approaches to division tasks. The students' strategies were coded according to the level of abstractness and degree of sophistication of their solutions, drawing on the categories of earlier studies (Heirdsfield et al., 1999; Mulligan & Mitchelmore, 1997). Abstractness used in this context refers to an ability to imagine the individual items as a composite unit and to solve a problem mentally without the use of physical objects (including fingers), drawings or tally marks. The strategies presented in Figure 1 are in a hypothetical order of sophistication moving from the use of concrete model, to partial abstraction using skip counting, repeated addition or subtraction, to total abstraction using building up, doubling or halving or multiplicative calculation.

Strategy	**Definition**
Direct Modelling (DM)	Uses sharing or one to many grouping with materials, fingers or drawings and calculates total by counting all, skip or additive counting.
Transitional Counting (TC)	Use of skip or double counting of the divisor to find the total. Some use of partial modelling with fingers or drawings using sharing or one-to-many grouping.
Repeated Addition (REA) or Repeated Subtraction (RS)	Repeatedly adds multiples of the divisor from zero until reaches the dividend, or subtracts multiples of the divisor from the dividend until reaches zero. Partial drawing/recording in some instances, if unable to fully coordinate the two composite units.
Building Up (BU)	Skip counts using the divisor up to the dividend, without the use of any drawing or tally marks.
Doubling and Halving (DH)	Derives solution using doubling or halving and estimation, attending to the divisor and dividend. Recognises multiplication and division as inverse operations.
Multiplicative Calculation (MC)	Automatically recalls known multiplication or division facts, or derives easily known multiplication and division facts, recognises multiplication and division as inverse operations.
Holistic Thinking (HT)	Treats the numbers as wholes—partitions numbers using distributive property, chunking, and or use of estimation.

Figure 1. Hypothetical Hierarchy of Solution Strategies for Whole Number Division Problems.

The strategies that precede Building Up all rely on some form of physical model or representation. Building Up and those that follow are referred to as abstracting or mental strategies as students can solve problems without the need to use any physical models or representations. The author classified the latter three strategies in Figure 1 as multiplicative because they reflect the simultaneous coordination of the multiplicand and multiplier in a situation that reflects an understanding the properties of multiplication and division and the inverse relationship between the operations.

Results

From the analysis of data two findings were evident. First, many more students than anticipated chose the challenge level of difficulty tasks rather than the medium or easy levels, and they responded correctly. Second, multiplicative strategies were the preferred strategy of choice to solve the problems at the challenge level of difficulty, as opposed to additive strategies (e.g., repeated addition or subtraction, skip counting).

Table 2 shows both the students' choice of task level of difficulty across the eight tasks and the number of correct responses for each of the partitive and quotitive whole number division problems. Of the 101 correct responses across the eight tasks, 73 were for the challenge level of difficulty and the remaining 28 were for the medium level of difficulty. The shaded columns indicate that the tasks in which students were unsuccessful. The 100% accuracy by those who chose the challenge level of difficulty indicates that these students were equally successful on both partitive and quotitive division across each of the semantic structures.

Table 2
Level of Difficulty Chosen and Correct Responses for Each Division Problem (n=13)

Task	EG-P	EG-Q	A/R-P	A/R-Q	RA-P	RA-Q	TM-P	TM-Q
Medium	4	3	4	3	3	5	3	3
Challenge	9	10	9	10	10	8	8	9
Total correct	13	13	13	13	13	13	11	12

Note: EG-P (Equal Group-Partitive), EG-Q (Equal Group -Quotitive); A/R-P (Allocation/Rate-Partitive); A/R-Q (Allocation/Rate-Partitive); RA-P (Rectangular Array -Partitive); RA-Q (Rectangular Array - Quotitive); TM-P (Times-as-Many-Partitive); TM-Q (Times-as-Many-Quotitive).

Table 3 shows the frequency of strategies on the different whole number division tasks for the challenge level of difficulty. In all but one instance (EQ-1) abstracting strategies were chosen across the eight tasks.

Table 3
Frequency of Strategy Choice for Each Semantic Structure

Semantic Structure	Task type	TC	BU	DH	MC	HT
Equal Groups	EG-P		1	1	5	2
	EG-Q	1	0	1	8	
Allocation/Rate	AR-P		1		8	
	AR-Q		5	1	4	
Rectangular Array	RA-P		1	1	7	1
	RA-Q				5	3
Times-as-Many	TM-P			3	3	2
	TM-Q		3	2	2	2

Note: TC (Transitional Counting); BU (Building Up); DH (Doubling/ Halving); MC (Multiplicative Calculation); HT (Holistic Thinking.

Four findings are apparent from the data presented in Table 3. First, there was little difference in students' strategy choice for partitive and quotitive tasks across the challenge level of difficulty or semantic structures. The one exception was the Allocation/Rate quotitive task which involved a money context and a divisor of 15. It could be argued that the size of the divisor may have prompted the use of a skip counting strategy. Second, multiplicative calculation was the most frequently chosen strategy across the eight tasks. This suggests having an understanding of multiplication supports students' development of division and facilitates their use of the inverse operation to solve division problems. Third, 86% of responses indicated the use of multiplicative strategies (DH, MC, HT) to solve the problems. Students who consistently used these strategies were thinking multiplicatively rather than additively. Fourth, there was little difference in students' strategy choice across the four semantic structures.

The following qualitative data exemplify the students' use of these strategies. To solve the Equal Groups quotitive challenge task these three students drew on their knowledge of multiplication facts as a starting point.

EG-Q: Seventy-two children compete in a sports carnival. Four children are in each event. How many events are there?

> Bindy: I started with 12 times 4 then doubled it to get 24 times 4 and that's 96, but that's too much. So I took away 20 from 96 and that gave me 76, and that's 19 times 4 but I still need to take away another 4 so it would be 18 events or 72 divided by 4 equals 18.

> Sandy: If it was 8 times 8, it's 64. So 9 times 8 is 72. So, if you've got 72 children, and 4 in each event you've got to halve the 8 and double the 9, so there are 18 events. So 72 divided by 4 is 18.

> Jules: I know 12 times 4 is 48, 20 times 4 is 80, but that's eight too much so it's 18, because you just take off 2 lots of 4 from 20. 72 divided by 4 is 18, so there are 18 events.

Although each of the students started with a known multiplication fact, subsequent strategies reveal different levels of thinking. Bindy used doubling, trial and error and subtraction; Sandy used doubling and halving; and Jules went to the nearest multiple of 10 and subtracted.

The following abridged excerpts from the interviews illustrate the students' use of multiplicative thinking for the Allocation/Rate challenge task.

A/R-Q: I have 90 cents to spend on stickers. If one packet of stickers costs 15 cents, how many packets of stickers can I buy?

> Jules: 6 packets. I know 4 times 15…equals 60 plus 30… equals 90. So 6 times 15 equals 90, or 90 divided by 6 equals 15.

> Bindy: Thinking…6. Double 15 is 30, that's 2 times, double 30 is 60 so that's 4 times and another 30 is 90 so that's 6 times. So 90 divided by 15 is 6, so it's 6 packets.

> Sandy: 3 times 15… equals 45. Double that is 90 so 6 times 15 is 90, so that's 6 packets of stickers. So 90 divided by 6 is 15.

> Mark: 1 packet costs 15 cents, two cost 30 cents, 45 is 3 times. 60, is 4, 75 is 5, 6 is 90, 'cause six times 15 equals 90, or 90 divided by 15 equals 6. That means you can get 6 packets for 90 cents.

In solving the above task, these students clearly indicated their preference for using multiplication. The size of the divisor/multiplicand (15) may have resulted in Jules, Bindy, and Sandy using more than one strategy, whereas Mark used rate type thinking.

For the partitive task the students tended to use the divisor as the multiplicand rather than the multiplier as illustrated in the following excerpts of students' responses from the interviews.

A/R-P I rode 63 kilometres in 7 hours. If I rode the same speed the whole way, how far did I ride in one hour?

> Jules: I know that ten times seven equals 70 and I have to take away 7 to get 63, so it's 9 kilometres in one hour. So 63 divided by 7 equals 9.

Sandy: If you were timesing by 7 you'd get to 63, because when you count by sevens 63 is nine sevens. So then you've ridden 9 kilometres. Another way to say it is 63 divided by 7 is 9.

Mark: Nine because I know 9 sevens is 63, so you rode 9 kilometres in an hour. So 63 divided by 7 is 9.

All three students appeared to have ignored that it was a partitive task and treated the divisor (7) as the multiplicand. Even though students used this thinking, their division number sentence was written correctly (e.g., 63÷7=9), indicating their understanding of the role of the divisor in the context of the problem.

Discussion and Conclusion

The findings relating to students' performance on and approaches to division word problems indicate there was little difference in students' performance or strategy choice on partitive and quotitive tasks across the four semantic structures. From this it is reasonable to infer that students at this level are capable of interpreting all four semantic structures and both forms of division.

The four findings relating to the students' solution strategies highlight the value of allowing students to choose the task level of difficulty in revealing students' untapped mathematical capabilities. Enabling students to engage with complex tasks prompts the use of more sophisticated strategies than may normally be the case. The excerpts of student responses indicate a multi-layering of strategies, which suggests that these students were engaging with the mathematics of the tasks at a level of sophistication not expected at Grade 3. Implicit in this is the idea that, when challenged, students are required to look for options other than those with which they are familiar. From this it is reasonable to infer that some Grade 3 students are potentially capable of more than the curriculum currently indicates they are able to achieve.

The students' preference for using multiplicative calculation was also evident in studies of students in Years 4 to 6 involving numbers beyond the multiplication fact range (Fischbein et al., 1985; Heirdsfield et al., 1999). It could be argued that students used multiplication intuitively as it was not directly taught to the students in the study as a way to solve division problems, nor was the use of doubling and halving. It also supports the case for teaching both operations together rather than separately.

The use of the divisor as the multiplicand when engaging with partitive tasks is a common strategy employed by students when solving partitive division problems, as reported by Murray et al. (1992). It could be argued that doing so reflects both an understanding of the inverse relationship between multiplication and division and an awareness of the relationship between dividend, divisor and the quotient, and the role of each in a division problem.

In an earlier paper (Downton, 2009), the author reported that there was little difference between the strategies students used for partitive and quotitive division word problems and suggested that the form of division is not an influential factor on their strategy choice. The argument being made in this present paper is not whether the problem is partitive of quotitive but on the students' intuitive use of multiplication or a multiplicative strategy to solve division problems, regardless of the form. As indicated by Squire and Bryant (2002) students need experiences with a range of problem types to develop their conceptual understanding of the multiplicative relationships inherent in a problem. However, underpinning these experiences must be the developing understanding of the relationship between the dividend, divisor and the quotient, and the roles of each within a division problem (Correa et al., 1998). The fact that the majority of students consistently used multiplication to solve division problems suggests that the operations should be taught together, as recommended by Neuman (1999). Doing so may support students understanding of multiplication and division as inverse operations.

The implications for mathematics instruction and teacher educator are twofold. First, to engage students in division word problems that incorporate number combinations that cannot be intuitively manipulated using additive thinking. It appears that by doing so can prompt the use of sophisticated strategies. Second, that developing a relationship between multiplication and division is more significant than differentiating between the types of division as evidenced by the fact that the students were equally successful on the partitive and quotitive tasks.

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