The Complexity of One-Step Equations

Bing Ngu
The University of New England
<bngu@une.edu.au >

An analysis of one-step equations from a cognitive load theory perspective uncovers variation within one-step equations. The complexity of one-step equations arises from the element interactivity across the operational and relational lines. The higher the number of operational and relational lines, the greater the complexity of the equations. Additionally, the presence of a special feature increases the complexity of the one-step equations when the number of operational and relational lines is kept constant.

Mathematics educators acknowledge the power of algebra in mathematical problem solving (Kieran, 1992; Stacey & MacGregor, 1999). Although the use of an equation to solve the unknown is central to an algebraic approach in problem solving, regrettably, inadequate attention has been devoted to help students master equation solving skills. In particular, for one-step equations which act as a transition from pre-algebra to algebra learning, mathematics educators may overlook its significance within the topic of equation solving.

On examining three mathematics textbooks currently used in Australia, we found that the balance method is the popular method for equation solving (Kalra & Stamell, 2005; Smith, Iampolski, Elms, Roland, & Rowland, 2011; Vincent, Price, Caruso, McNamara, & Tynan, 2011). An analysis of the one-step equations in the three textbooks reveals that the classification of "one-step" reflects the manner in which the solution procedure is portrayed to the learners. For example, when solving one-step equation such as $x+5=13$, the critical procedural step (one operational line) is to perform –5 on both sides to alter the problem state of the equation and yet at the same time to maintain the equality of the equation. In essence, the critical procedural step acts as a point of reference to classify one-step equations.

Since one-step equations involve one critical procedural step, students tend to use trial and error or substitution method to solve one-step equations. It is not surprising that mathematics educators may erroneously think that one-step equations are simple and should be easy for students to grasp and learn. Nonetheless, inadequate knowledge of solving one-step equations not only will adversely affect the use of the algebraic approach in problem solving but also it will have a negative impact on the development of equation solving skills for equations involving multiple steps (e.g., two-step and multi-step equations). For example, without a strong foundation of using a formal method (balance method) to solve one-step equations, students will find it difficult to learn multi-step equations such as $5x-7=3x+4$ where the use of trial and error or substitution will become cumbersome and practically not feasible. Therefore, a renewed focus to classify one-step equations would provide a framework for both mathematics educators and researchers to put one-step equations in the agenda of the mathematics curriculum.

Based on a structural analysis of 1097 algebra story problems, Mayer (1981) developed a framework for classifying algebra story problems: (1) families (source of formula), (2) categories (story line), and (3) templates (a specific propositional structure). Classifying algebra story problems in this manner enables mathematics educators and researchers to teach or research a specific type of algebra story problems. For example, the identification

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of different templates embedded within the '*discount*' problems enables the teaching and learning of *discount* problems to occur at the template level. Note that the solution procedure of one template differs from another template.

Our aim in the present study is to analyse the complexity of one-step equations. Similar to Mayer's study (1981) in the context of algebra story problems, our goal is to classify one-step equations so as to provide a framework for mathematics educators and researchers to teach and research one-step equations. We classify one-step equations from a cognitive load perspective.

Cognitive load theory (Sweller, 2011, 2012; Sweller, van Merrienboer, & Paas, 1998) has evolved to be a prominent theory in instructional designs since its inception in the 1980s. Central to cognitive load theory is a limited working memory where our cognitive activities take place, and an unlimited long-term memory which stores a huge number of schemas. Cognitive load theory suggests that learning needs to build upon our prior knowledge (schema) in the long-term memory and learning will be hindered if we overload our working memory with irrelevant cognitive activities.

The complexity of learning tasks constitutes the intrinsic cognitive load. The degree of the intrinsic cognitive load is proportionate to the degree of the element interactivity. Element interactivity arises from the interaction between multiple elements within a learning task. An element is anything that needs to be learned.

Learning tasks can be classified as low element interactivity tasks or high element interactivity tasks. In a mathematics domain, an example of a low element interactivity task is learning the length (or breadth) of a table. The length or breadth is considered as one element only and each can be learned independently of the other. Because the learning of each element can occur in a serial manner, it is unlikely to put a strain on the working memory. In other words, low element interactivity tasks impose low intrinsic cognitive load. In contrast, learning to calculate the area of the table $A = l \times b$ constitutes a high element interactivity task. There are three elements in the area concept (*A, l, b*), and the learner cannot learn each element in isolation. Rather, the leaner needs to learn the interaction (multiplicative relation) between *l* and *b,* giving rise to *A.* According to cognitive load theory*,* working memory devotes a high cognitive load to manipulate multiple interacting elements simultaneously.

In mathematics learning, one often uses the term *understanding of mathematical concepts*. Based on cognitive load theory, understanding occurs when the learner understands the relation between multiple elements. This is often the case with high element interactivity tasks. However, if there is no relation between elements, then, understanding does not apply. This is often the case with low element interactivity tasks. The focus of this present study was to uncover the complexity of one-step equations from an element interactivity perspective. We will begin with Type 1 of one-step equations having one operational and two relational lines.

Type 1 of One-Step Equations: 1 Operational and 2 Relational Lines

We use relational and operational lines to describe the solution procedure of equation solving. An operational line requires the learner to perform the same operation (e.g., –5 on both sides) to alter the problem state of the equation and yet at the same time to preserve the equality of the equation. In contrast, a relational line shows a quantitative relation between elements so that the left side equals to the right side. We will describe the solution procedure involved in solving $x + 5 = 13$ via the balance method from an element interactivity perspective below.

Line 1	$x+5=13$	
Line 2	–5 –5	(–5 on both sides)
Line 3	$x=8$	

Figure 1: Type 1 of one-step equation

As depicted in Figure 1 (see also Table 1), line 1 is a relational line which comprises three elements (x, $+5$, 13) and three concepts: (1) $x+5=13$ is an algebraic sentence in which the x (pronumeral) can be replaced by a number so that the left side equals to the right side of the equation, (2) the equal sign (=) represents a quantitative relation in which the elements between the left side and right side are equal, (3) to find x, the learner needs to perform the same operation on both sides (what is done on the left side should also be done on the right side) to balance the equation. The learner needs to assimilate the interaction between these three elements and concepts simultaneously. Line 2 is an operational line and it has one element (–5). The learner performs an operation (–5) on both sides (cancel –5 with +5 on the left side, the same –5 must be done on the right side) to maintain the equivalence of the equation: (1) Here, element interactivity occurs at both sides of the equation. The learner not only needs to process –5 with +5 on the left side, but also –5 with +13 in the right side. Line 3 is a relational line and it has two elements (x, 8). If the learner can process the element interactivity involved in lines 1 and 2 successfully, then x equals to 8 being the solution would seem straightforward.

Table 1 displays different types of Type 1 of one-step equations. Using the number of operational and relational lines as a point of reference, the number of interacting elements and thus the degree of element interactivity of all Type 1 of one-step equations is the same regardless whether the equations involve a special feature or not (Table 1). Nonetheless, those equations without a special feature would be easier to learn than those equations with a special feature. We will explain our assumption as below.

Type 1 of One-Step Equations

Positive numbers – As shown in Table 1, both equations such as $-2x\times-5x$ and $4n=16$ do not have a special feature. These equations involve the manipulation of positive numbers (addition/subtraction, multiplication/division). We expect students to have adequate prior knowledge of operating positive numbers and thus these equations should be easy for them to learn.

Type 1 of One-Step Equations with a Special Feature

Negative numbers – The equation such as $20\%x=100$ involves negative numbers. Manipulating negative numbers is a challenge for both students (Ayres, 2001) as well as adults (Das, LeFevre, & Penner-Wilger, 2010). In Ayer's (2001) study, students recorded high task difficulty (cognitive load) and tended to made error when they were asked to multiply two negative algebraic expressions such as $-2x\times-5x$.

A decimal/percentage – The equation such as $x/0.5=9$ is analogous to an equation such as $y/3=2$ because there is one-to-one match in elements (e.g., 9 matches with 2). Similarly, the equation such as $10\%x=40$ is analogous to an equation such as $2x=6$. Nonetheless, equations involving a decimal or a percentage may pose a challenge for students due to their failure to connect percentage, decimal and whole number (Parker & Leinhardt, 1995).

Table 1
Type 1 of One-Step Equations (1 Operational Line and 2 Relational Lines)

Equation type	Solution procedure
Type 1 of one-step equations	
Positive numbers	$x+5=13$ $-5 \quad -5$ $x=8$
Positive numbers	$4n=16$ $\div(4) \quad \div(4)$ $n=4$
Type 1 of one-step equations with a special feature	
Negative numbers	$a-3=-7$ $+3 \quad +3$ $a=-4$
A decimal number	$\frac{x}{0.5}=9$ $\times 0.5 \quad \times 0.5$ $x=4.5$
A percentage	$10\%x=40$ $\div 10\% \quad \div 10\%$ $x=400$
The solution is a fraction or a decimal	$3m=2$ $\div 3 \quad \div 3$ $m=2/3$
Pronumeral on right-side	$1=2p$ $\div 2 \quad \div 2$ $0.5=p$

Note: All Type 1 of one-step equations share the same one operational lines and two relational lines and thus they share the same degree of element interactivity. However, those equations without a special feature would be easier to solve than those equation with a special feature.

The solution involves a fraction or decimal – It is no long sufficient for students to rely on trial and error or substitution method to solve equation such as $3m=2$ because the solution involves a fraction or a decimal. Therefore, students may fail to solve such equation unless they are competent at using a formal method (e.g., balance method) to solve the equation.

Pronumeral on the right side – The equation such as $1 = 2p$ has the pronumeral located on the right side instead of the normal left side of the equation. Because of this, students have to work from right side to left side in reverse order, which may affect their ability to solve the equation. It should be stressed that this type of equation is significant because it indirectly tests students' grasp of the 'equal sign' concept irrespective whether they work from left side to the right side or vice versa.

Apart from Type 1 of one-step equations having one operating line and two relational lines, there are also Type 2 of one-step equations having two operating lines and three relational lines. Type 2 of one-step equations will be discussed next.

Type 2 of One-Step Equations: 2 Operational and 3 Relational Lines

Table 2 shows the solution procedure of Type 2 of one-step equations involving two operational and three relational lines. The element interactivity associated with Type 2 of one-step equations will be higher than Type 1 of one-step equation depicted in Figure 1 owing to the combined increase in the operational line (2 vs. 1) as well as relational line (2 vs. 3). Because students need to manipulate the interaction between elements across the operational and relational lines, the higher the number of operational and relational lines, the higher the cognitive load. In other words, the degree of element interactivity arises from the operational and relational lines serves as a point of reference to determine the complexity of one-step equations. As shown in Table 2, Type 2 of one-step equations comprises those equations with and without a special feature. Type 2 of one-step equations will be discussed next.

Type 2 of One-Step Equations

Negative pronumeral – Of particular interest is the Type 2 of one-step equations with a negative pronumeral (Table 2). Using trial and error or substitution method, the equation with a negative pronumeral such as $5 - n = 0$ poses no greater challenge than $5 + n = 0$. Nonetheless, if the learners are not taught properly how to solve this type of equation using a formal method, then a problem may arise when they learn more difficult equations involving multiple steps (e.g., $(3 - p)/4 = 7$.

Pronumeral as a denominator – Apart from those equations with a negative pronumeral, the equation $4/a = 2$ in which the denominator acts as a pronumeral may also present a challenge to students. Again, the use of a short-cut method (e.g., substitution method) on this type of equation may negatively impact on students' ability to solve trigonometry problems such as $5/x = \sin 30^{\circ}$. Hence, it is critical that mathematics educators need to be aware of the degree of element interactivity involved and spend more time on this type of equation which is connected to trigonometry problems.

Type 2 of One-Step Equations with a Special Feature

Negative pronumeral plus negative number – Students not only have to manipulate negative pronumeral but also negative number for the equation such as $-4-y=5$. Therefore, such type of equation will be more complex as compared to those equations with a negative pronumeral.

Pronumeral as a denominator and is located on right-side – Again, the equation $2=8/n$ in which the pronumeral acts as a denominator and is located on the right-side is linked to the topic of trigonometry problems. Thus it is important to ensure that students are able to solve this type of equation so that they can transfer the knowledge to solve equivalent trigonometry problems in the future.

Table 2
Type 2 of One-Step Equations (2 Operational Lines and 3 Relational Lines)

Equation type	Solution procedure
Type 2 of one-step equations	
Negative pronumeral	$5-n=0$ $-5 \quad -5$ $-n=-5$ $\div(-1) \quad \div(-1)$ $n=5$
Pronumeral as a denominator	$\frac{4}{a}=2$ $\times a \quad \times a$ $4=2a$ $\div 2 \quad \div 2$ $2=a$
Type 2 of one-step equations with a special feature	
Negative pronumeral plus negative number	$-4-y=5$ $+4 \quad +4$ $-y=5$ $\div(-1) \quad \div(-1)$ $y=-5$
Pronumeral as a denominator and is located on right-side	$2=\frac{8}{n}$ $\times n \quad \times n$ $2n=8$ $\div 2 \quad \div 2$ $n=4$

Note: All Type 2 of one-step equations share the same two operational lines and three relational lines and thus they share the same degree of element interactivity. However, those equations without a special feature would be easier to solve than those equations with a special feature.

Teaching and Learning One-Step Equations

The classification of one-step equations provides a framework for mathematics educators to teach one-step equations in a hierarchical level of complexity. Obviously, mathematics educators need to begin with Type 1 of one-step equations having one operational and two relational lines because they impose the lowest degree of element interactivity. Within Type 1 of one-step equations, it is advisable to teach those equations without a special feature, followed by those equations that involve a special feature. Mathematics educators need to spend adequate time on a specific type of Type 1 of one-step equations (e.g., a decimal/percentage) to allow students to master skills in solving different types of Type 1 of one-step equations. In addition, it is important to highlight the connection between different topics of mathematics when teaching Type 1 of one-step equations. For example, learning how to solve the equation such as $10\%x = 40$ is likely to pave the way for students to use an algebraic approach to solve real-life problems such as " *If 25% of my weekly salary is $100, what is my weekly salary*?" in the future.

Once students have acquired skills to solve Type 1 of one-step equations, mathematics educators can expose them to Type 2 of one-step equations which incur a higher degree of element interactivity than the former. Once again, the learning of Type 2 of one-step equations should start with those equations without a special feature, followed by those equations with a special feature. Again, mathematics educators need to emphasise the connection between equation solving and other mathematics topics. For example, highlighting the link between those equations in which the pronumeral acts as a denominator and trigonometry problems will encourage students to engage in mathematical thinking and reasoning skills between different mathematics topics.

Researching One-Step Equations

The classification of one-step equations has uncovered the complexity of one-step equations. It would be of interest for researchers to investigate: (1) whether Type 1 of one-step equations are indeed easier than Type 2 of one-step equations for students to learn, (2) whether those equations (Type 1 or Type 2 of one-step equations) without a special feature would be easier to learn than those equations with a special feature, (3) what is the best way to teach equations involving a decimal, a percentage, and a fraction? (4) how to help students learn those equations with the pronumeral on the right side? (5) why is it a challenge to students when the solution of an equation involves a decimal or a fraction?

Discussion

Cognitive load theory has shed light on the complexity of one-step equations. Relying on the critical procedural step (operational line) alone is inadequate to determine the complexity of one-step equations. Rather, the use of the number of operational and relational line reflects the degree of element interactivity, which in turn, illuminate the complexity of one-step equations in a hierarchical order. Hence, one-step equations comprise Types 1 and 2 equations. Type 1 of one-step equations (1 operational and 2 relational lines) are easier to learn than Type 2 of one-step equations (2 operational and 3 relational lines) because the former impose a lower cognitive load than the latter.

Although the degree of element interactivity remains constant for all types of equations within Type 1 or Type 2 equations, those equations without a special feature are easier to learn than those equations with a special feature. Therefore, it is critical that the teaching and learning of one-step equations begins with Type 1 equations (without a special feature followed by a special feature), and Type 2 equations (without a special feature followed by a special feature).

More importantly, mathematics educators need to emphasise equation solving skills and the preparation for students to engage in an algebraic approach in problem solving in the future. For example, proficiency in equation solving skills may help students appreciate the power of setting up an equation to solve real-life percentage problems such as $20\%x = 100$.

The classification of one-step equations also provides a framework for researchers to investigate ways to improve students' equation solving skills particularly on different types of one-step equations which have traditionally been overlooked.

To conclude, this study has pointed out the need to bring one-step equations into the agenda of mathematics curriculum. It is necessary to strengthen students' equation solving skills so that they will be more likely to succeed in using an algebraic approach in problem solving.

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