

The Language Used to Articulate Content as an Aspect of Pedagogical Content Knowledge

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Mathematical knowledge in classrooms is mediated through the use of both technical and informal language. This paper is a report of a study of the language use of teachers as they examine students' work and discuss teaching for the topic of fraction operations. This provides a window on their pedagogical content knowledge and also on the way in which language is used to make sense of mathematical knowledge, either personally or for students. It was found that some mathematical knowledge appeared to be *taken as understood*, perhaps because the expected words were used.

Introduction and Background Literature

Recently the author was reading some material that reported on teachers' solutions to a particular class of proportion problems. What was striking about these solutions, which came from a country different from her own, was the consistency of the terminology and representations used by the teachers as they presented their mathematical reasoning. In addition, this consistency of language seemed to give the teachers a greater capacity to articulate the mathematics: they were fluent in the way they used the terminology in support of their solutions. Their content knowledge of mathematics appeared to be enhanced by this precision of language, and the language was used effectively in support of their further discussions about how they might help students with similar problems. This suggests that the terminology and representations used by teachers might give insights into their pedagogical content knowledge (PCK).

There are a number of frameworks associated with the knowledge for teaching mathematics, and they categorise that knowledge in slightly different ways. The knowledge quartet of Rowland, Huckstep, and Thwaites (2005) is a little different from some of the other frameworks, in that it views certain aspects of knowledge dynamically. The more static component—foundation—includes “the meanings and descriptions of relevant mathematical concepts” (p. 265), which includes relevant terminology. One of the more dynamic knowledge-in-action aspects—transformation—also makes reference to language, and considers how “the teacher's own meanings and descriptions are transformed and presented in ways designed to enable students to learn it” (p. 265), with the acquisition of essential vocabulary explicitly mentioned as part of the teacher's work. The Hill, Ball, and Schilling framework (2008, p. 377) for mathematical knowledge for teaching highlights common content knowledge, which includes terminology, although it might be argued that some terminology (e.g., the use of the word *whole* with respect to fractions) is really only used by teachers, rather than being in general mathematical use. Finally, the PCK framework of Chick, Baker, Pham, and Cheng (2006) identifies at least two components that allude to language use: *knowledge of representations* (which might be construed to include language, since language is used to signify and represent ideas), and *knowledge of explanations*, which requires use of appropriate language.

Boero, Douek, and Ferrari (2002) wrote about the role of natural and symbolic languages in mathematics, and assert that “only if students reach a sufficient level of familiarity with the use of natural language in the proposed mathematical activities can

they perform in a satisfactory way” (p. 242). They also highlighted the teachers’ role in increasing students’ linguistic competencies, including in discussing solutions. Boero et al. discussed language as a mediator between mathematical objects, properties, and concepts and the development of *theoretical systems*, by which they seem to mean a connected conceptualisation of bigger mathematical ideas (such as *rational numbers and operations* in the case of this paper). In addition, they highlighted the role of language as a tool for the validation of statements.

The area of fractions is one of the first mathematical topics that moves beyond the concrete arena of natural numbers, where operations are readily visualised and described, often with words that are part of everyday language. With fractions come new technical words, such as *numerator*. The conceptualisations of part-whole relationships and operations such as addition and division must be mediated through language use. In their seminal chapter on rational numbers Behr, Harel, Post, and Lesh (1992) explore the complexity of the domain, and highlight the idea of a *unit*”. When a fraction is defined in relation to a whole, it requires the conceptualisation of new units, namely the individual parts of the whole determined by the denominator. Thus, in the fraction $\frac{4}{5}$, we must think of $\frac{1}{5}$ as the unit (determined by creating 5 equal pieces from the whole), and consider 4 of these units. Simultaneously interpreting the information supplied by the denominator and the numerator is required in order to see a fraction as a single quantity. A complete understanding of the domain of rational numbers requires, among other things, understanding of how fractional quantities operate on other quantities and how we compute efficiently with such quantities (including finding valid algorithms).

Ma (1999) found that primary teachers from the United States (US) and China varied in their capacity to make sense of, for example, fraction division. In giving explanations of their reasoning, the Chinese teachers seemed to have consistent terminology with which to refer to the components, operations, and procedures (e.g., referring to *quotient* and *dividing by a number is equivalent to multiplying by the reciprocal*), whereas the US teachers were less consistent and, indeed, unsure about the language (e.g., *change them into sync, flip over and multiply*). In addition, the Chinese teachers could more often provide mathematical reasons to justify the processes used.

With this in mind the present research examines the language use and mathematical reasoning made evident in teachers’ discussions of students’ work with fractions. Specifically, it looks at the consistency of language and the way in which the mathematical ideas inherent in the terms and operations were discussed and justified.

Method

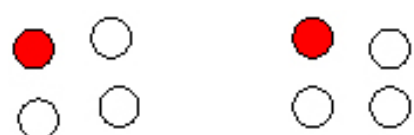
Data were gathered during focus group discussions involving some invited experienced teachers and the researchers (including the author, and colleagues from the *Powerful Knowledge* project of which this study was a part). There were primary and secondary focus groups sessions with around four to five teachers each in both Tasmania (Tas) and New Zealand (NZ), and a session with four primary teachers in Victoria (Vic). The teachers were purposely selected, and were known to the researchers as having an interest in mathematics teaching. The focus groups were intended to explore the knowledge that is brought to bear in teaching mathematics, and to provide an opportunity to access the tacit and explicit knowledge on which teachers draw in the act of teaching.

The stimuli for the focus group discussions were items covering a range of school level topics and pedagogical content knowledge issues. The researchers presented items in turn, and discussion ensued about the nature of and responses to the situation. The intent was to

explore the kind of knowledge that is needed for these situations, rather than to gauge any or all of the teachers' capacities to respond. Because the nature of knowledge for teaching mathematics was the focus of the data gathering, the researchers themselves were not independent of the discussion and contributed to the conversations and, thus, to the data. Discussions were audio-recorded and later transcribed.

Two of the items used with the focus groups have been selected for analysis for this study, both coming from the domain of fraction operations. The task shown in Figure 1 was presented to the New Zealand and Tasmanian primary teachers only, and concerned a misconception that is reinforced by the child's choice of representation. To determine an appropriate response to the student's ideas, the teacher might choose to draw on other alternative representations (as suggested by some of the options in the question in Figure 1), or may choose to work with the child's self-selected representation. There are advantages and disadvantages associated with each of these approaches (see Chick, 2011).

A student says that $1/4 + 1/4$ is $2/8$. She uses counters to show this as follows:



Given what the student has just shown you, which of the following representations of $1/4 + 1/4$ is most likely to help her to see that $1/4 + 1/4 = 1/2$?

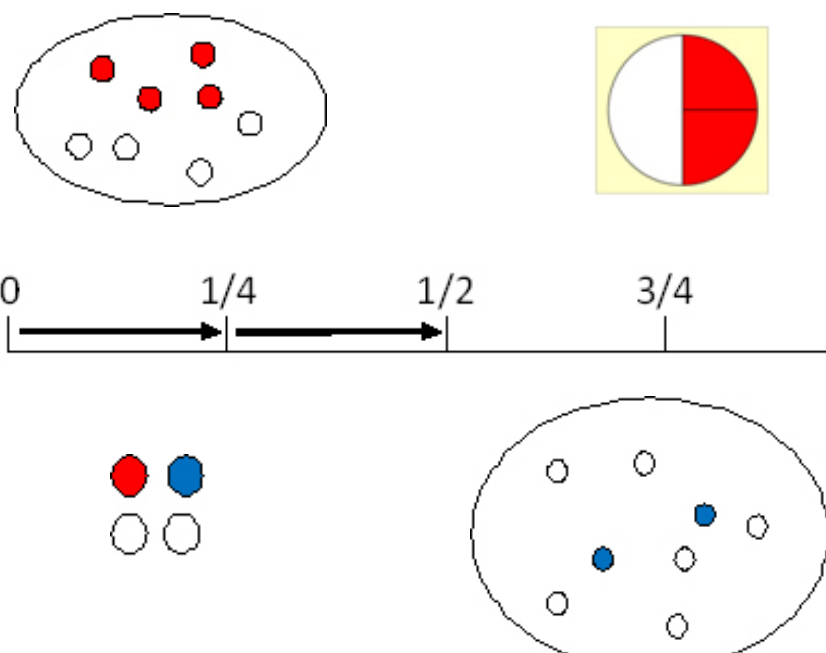


Figure 1. The modelling fraction addition item.

The second item, shown in Figure 2, was presented to the secondary and Victorian primary groups, and involved a student's computation of the quotient of two fractions, obtained by using an alternative to the standard *invert-and-multiply* algorithm.

When asked to describe how they determined $\frac{2}{3} \div \frac{3}{4}$	$\frac{2}{3} \div \frac{3}{4}$
a student wrote the following on the classroom whiteboard:	$= \frac{8}{12} \div \frac{9}{12}$
	$= 8 \div 9$
	$= \frac{8}{9}$

Figure 2. The division of fractions item.

The transcripts of the focus group interactions were examined to investigate the mathematical language used when discussing the issues associated with these teaching situations. In particular, the key foci for the analysis were:

- commonalities and differences in language across individuals;
- strengths and inadequacies in language use;
- the extent to which language aligned well with the content under discussion; and
- the extent to which language was used to address mathematical issues implicitly and explicitly.

Results

Modelling Fraction Addition

One of the key issues underpinning the fraction addition situation is the identification of the whole. This terminology came up regularly in conversation, almost always as *whole* but once as *unit of analysis*.

... if we've already talked about our understanding of fractions is, "How many equal parts make that whole, so how many do you need to make that whole? And what part of our fraction tells us how many equal parts make the whole?" [NZ, Primary]

I like that we have to move away from counters because in some ways the counters are talking ... what's the unit of analysis. [Tas, Primary]

In the situation illustrated in Figure 1, the student does not appear to have misconceptions associated with a specific fraction of a given whole, although there was some initial doubt about this in one focus group, after one teacher had suggested that the student needed more experience with the idea of a "quarter of a group". Another teacher counter-argued that the student's basic conception of a quarter was sound, saying "she has actually represented ... one counter out of four ... she sees that one out of four is ... it's working with four counters as a whole, and here's the one". The specific issue associated with the misconception illustrated in Figure 1 is that, in order to be added, the two quarters and their sum must be represented and conceptualised in reference to the same whole. This was not always clearly articulated among the teachers who considered the situation, and the first extract below actually preceded the push to focus on the basic conceptualisations, with the concluding comment about there being two groups not followed up at first.

She's got two wholes here ... but I think I'd have to go right back to the beginning of fractions with her ... yeah, forget about the adding, go back to a quarter of that, of one, and then a quarter of another group and so that you can transfer the quarter of the group to here, and realise it's a quarter of another group of what she's done. [Tas, Primary]

She doesn't catch on that four in one quarter and eight in one-eighth is [in reference to] the same one whole. [Tas, Primary]

The second quote above, which arose later in the conversation among the Tasmanian teachers, gets to the heart of the matter but without specifying that the problem of the identification of the whole even applies with the original two quarters which were not shown with respect to a common whole.

Although it was intended that the focus group discussion should address Figure 1 (while also allowing this to be a springboard for other discussion), the New Zealand focus group spent some time discussing teachers' experiences with their own students' work. In so doing, they discussed aspects of the role of the whole. For example, one teacher told of the way students discussed the equivalence of mixed and improper fractions, and another described a student's approach to showing that one-third cannot be the same as three-eighths. In these accounts, the discussion of the whole and fractions was fluent and correct. The problematic nature of the whole in Figure 1 was hinted at only once, however, in the following.

If a child said to me, "Well, here's my whole" ... you know, but then you'd be saying, "Well, what fraction is that? If that's your whole, what's your fraction?" You know, "What's the fraction?" so, "Really is it a quarter plus a quarter?" ... So it's making sure that they really understand, that you're on the same wavelength in what you each consider the whole to be. [NZ, Primary]

In this case, if the words *are* referring to the need for a consistent whole—and it is not entirely clear that they are—then here the language has not been explicit about the mathematical details of the problem. However, two of the other teachers concurred with the statement, suggesting that some aspect of this contribution was understood and received agreement. This hints at some *taken as understood* common understanding held by the teachers, but the lack of clarity in the language begs the question of whether or not the *same* understandings were actually held.

Division of Fractions

In the division of fractions scenario shown in Figure 2, the student's computation resembles aspects of the addition algorithm, in that a common denominator and resulting equivalent fractions are found for the two fractions prior to continuing with the division process. The final answer is correct; the issue is whether or not the student's method is valid, and, if so, on what grounds.

On seeing the student's solution, one of the Victorian primary teachers seemed to be distracted by the seemingly incorrect use of common denominators.

Very common, see it all the time, a mixture of algorithms that they've learnt, looks like the algorithm of changing the denominators, and then [indiscernible] algorithms, numerator to the numerator, denominator to the denominator, so it's just a mix up of things that they've got in their head going on, and they're just applying them ... randomly. [Vic, Primary]

Here the familiar aspects of one algorithm in a context different from its usual application appeared to lead to an assumption that the student's work is incorrect. The phrase "the algorithm" may suggest the teacher believed in an *authorised* way of doing things (this is speculative, but it is known that some people view the *standard* algorithm as the *correct* way of solving a problem). There was no explicit use of language associated

with *equivalent fractions* although this idea is, perhaps, implicit in the phrase “changing the denominators”. The use of the words “numerator” and “denominator” in the latter part of the quote seemed to be in reference to actions/changes that are not clearly specified; the use of the word “randomly” at the end suggests that the teacher thought that the student’s actions are muddled rather than purposeful. There was no disagreement with this interpretation from the other teachers in the group, and the next phase of discussion turned to whether or not fraction division was part of the primary school curriculum.

The use of a common denominator was also disconcerting for the Tasmanian secondary teachers, with one saying “It looks like to be they’re confusing it with the addition algorithm,” but some at least recognised that the answer was correct. As the following extracts show, the teachers wanted to have the student explain the thinking behind the solution in order to discern the student’s rationale. It is, however, ambiguous as to whether or not any of the teachers thought the approach was valid and applicable more generally. Although the words *numerator*, *denominator*, *common denominator*, and *algorithm*, were used correctly in general, they seemed to be used superficially and the deeper meanings were not considered in any attempt to explain and justify the student’s valid approach.

Well, firstly, I would ask them why they did what they did. You know, I think that’s really important that kids understand how to do something, why it works and when you use it. And in this case I am intrigued that they’ve got the right solution by not the standard algorithm so I would ask them why they did it like that ... and then engage them in a conversation about, “Let’s weigh up some of the advantages and disadvantages of the different strategies you’re looking at now.” And see if they would actually change their mind about what they’ve done there. [Tas, Secondary]

Later in the discussion this teacher was able to articulate more about what was going on mathematically, but there was still much that was implicit in the explanation.

If this kid knew why they were doing that and they explained it in a way that makes sense to me, I want to add that to my library because in many ways it makes more sense than what we teach them because it is linked and connected to the addition and the subtraction one quite well because if you actually do the eight divided by nine and the twelve divided by twelve ... so you’re nearly there. [Tas, Secondary]

The first of the following teachers suggested that the student thinks you can “lose the twelves”, but did not examine if there are mathematical reasons that make this a valid step. The vanishing twelves, from the second line of the student’s solution to the third, was a point of concern for the other teachers, too, expressed in different ways.

You’d still have to ask why because maybe that’s not ... there’s something else is going on, that you can go about, you know, maybe they just noticed that you lose the twelves or something when they’ve been doing it elsewhere. But you really have to ask them first what’s happening. ... If they continued on the ‘correct’ algorithm, putting over the common denominator wouldn’t have mattered. ... I mean that’s the thing. I mean it wasn’t needed, but wasn’t incorrect, but there’s some reason they seem to think somehow you, once you put them over the common denominators you can forget the denominator. [Tas, Secondary]

When I got through to here, I thought, “Oh! Clever kid!” like, you know how when we’re teaching, we’re teaching the algorithms and they’ve got to remember, “Okay, when I divide fractions, what do I do again?” and the recall of that. I find that in adding fractions they get under control putting over common denominators fairly well and I think they do that a lot in primary school too before they come into grade 7. I think this is really clever in that again, we’re using the same sort of thought pattern of putting it over a common denominator again. Where I would be asking questions, even though I’d still ask ... why did you do that, like how did you get from at the second line to the third line? What happens there? [Tas, Secondary]

Yeah, assuming like that's over one, or we're having it over the same denominator means that they magically disappear or- that there is a lack of reasoning between those two steps. [Tas, Secondary]

The disappearing twelves can, in fact, be explained intuitively by realising that twelfths are the units for each fraction. Since each fraction is expressed in terms of the same unit (as shown by the same denominators), then the quotient is obtained from the quotient of the number of units in each of the dividend and divisor (i.e., 8 and 9, respectively).

The New Zealand secondary group also discussed asking the student to explain why he/she changed the fractions to equivalent fractions with a common denominator, with at least one misled by this out-of-place application of part of the addition algorithm. A few minutes later one of the teachers claimed that the solution was “Absolutely right. It’s a perfectly valid way to do it.” One of the teachers had had a student take a similar approach in class, and there was some extended discussion about the applicability of the method.

T1: I can't remember whether the student actually had the understanding or whether they just ... but they were definitely using their prior knowledge of adding fractions. ... I did a little bit more work on it myself, and um, I do use it sometimes, but it's a bit limited in its use, because the numbers have to work, but it is a valid method that's come out. ... So using equivalent fractions, and then, ... I think the student was saying, I think, ... [they] were going eight divided by nine is, is, yeah, to get the eight divide, and twelve divided by twelve is one ... However, it works, and I think when I did some more work on it myself, it does work, but you can get into a bit of a tangle.

T2: So it works for every whole [sic] number.

T1: Um, I don't know. I think ...

T2: I can't see why it wouldn't. ... It's just an algorithm. ...

Note that the word *algorithm* seems to have been endowed with an authority that implies that it makes things work, rather than as something that requires validation. Later the researcher interviewer tried to probe the deeper mathematical reasoning (“If we think of the multiplication algorithm for fractions, we multiply numerators, we multiply denominators ... So, if we're dividing two fractions, why not ... divide the numerators, divide the denominators?”) which was followed by some general discussion about setting up class explorations of the phenomenon, but again the discussion seemed to tacitly assume or agree with the method without confirming its validity.

Discussion and Conclusions

The teachers, for the most part, appeared to have a shared vocabulary with respect to basic fraction numeration and operations. There was some informality associated with the vocabulary on occasions (e.g., the phrase “numerator to the numerator, denominator to the denominator” used by one of the primary teachers, and “putting over a common denominator” by the secondary teachers), although there were no egregious errors of terminology. At times, however, some of the usage seemed to be *taken as understood*, in that teachers used some expressions that had the potential to be ambiguous in meaning, but there was an apparent assumption that they all understood what was meant by the terms (e.g., exactly which whole was meant by *whole* in the addition problem, and no one felt the need to question whether or not there was a reason behind the *loss* or *forgetting* of the denominator in the division problem).

There was very little detailed articulation of the fundamental principles underpinning the students' thinking. It may be that this assumption of shared meanings inhibits explicit examination of the underlying definitional mathematical meanings and implications. This

is not to say that single words that capture complex ideas are not powerful, but our familiarity with them might make it difficult to get back to and express the component foundational principles, and it is these that may be necessary in order to make sense of students' work and help them develop better understanding. For example, it was striking that there was little talk about the way that the numerator and denominator quantify the value of the fraction, by defining the size of the unit components (as determined by the denominator) and the number of such units being considered (as given by the denominator). The teachers seemed to understand fractions and the meaning of the numerator and denominator, but did not ever articulate that meaning explicitly.

In the discussion of the division problem, the meaning of *division* was completely taken as understood; the focus of the discussion was on computational operations and what could and could not be done with the fraction components and the operations, rather than on what it might mean for one fraction to be divided by another. It also seemed to be taken as understood—perhaps because the student *had* obtained the correct answer—that $(a \div b) \div (c \div d)$ is equivalent to $(a \div c) \div (b \div d)$.

The mixed results seem to suggest that perhaps it is time to talk about the way we talk, and the language we use in teaching mathematics. When working with pre-service teachers, it may be that we assume some things are taken as understood. Maybe, as a consequence, we are not as precise about defining and being consistent with the language we model, nor give enough emphasis to how that language allows us to discuss mathematical meanings, nor appreciate that these meanings allow us to justify mathematical procedures.

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References

- Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.
- Boero, P., Douek, N., & Ferrari, P. (2002). Developing mastery of natural language: Approaches to theoretical aspects of mathematics. In L. English (Ed.), *Handbook of International Research in Mathematics Education* (pp.241-267). London: Lawrence Erlbaum.
- Chick, H. L. (2011). God-like educators in a fallen world. *Proceedings of the 2011 annual conference of the Australian Association for Research in Education* (Hobart, TAS, 28 Nov – 1 Dec, 2011). Retrieved from: <http://www.aare.edu.au/>.
- Chick, H. L., Baker, M., Pham, T., & Cheng, H. (2006). Aspects of teachers' pedagogical content knowledge for decimals. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297-304). Prague: PME.
- Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39, 372-400.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, 8, 255-281.