Visualisation and Analytic Strategies for Anticipating the Folding of Nets

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Visual and analytic strategies are features of students' schemes for spatial tasks. The strategies used by six students to anticipate the folding of nets were investigated. Evidence suggested that visual and analytic strategies were strongly connected in competent performance.

Nets as an Opportunity for Spatial Reasoning

The importance of spatial reasoning to student success in mathematics and other STEM related fields is accepted (Okamoto, Kotsopoulos, McGarvey, & Hallowell, 2015; Sinclair & Bruce, 2015; Tosto et al., 2014). Given such significance some researchers lament the lack of focus on spatial reasoning globally, particularly in geometry curricula (Kell & Lubinski, 2013; Presmeg, 2008). Two other compelling reasons to promote spatial reasoning are the increased use of graphical representation to convey information across disciplinary fields (Lowrie, Diezmann, & Logan, 2012) and power of visual representations as tools for 'amplifying cognition' (Card & MacKinlay, 1999). One opportunity for students to exercise their spatial abilities is through working with nets, the 2-dimensional patterns that fold to form 3-dimensional solids (Cohen, 2003). The following content descriptor from Year 5 of the Australian Curriculum (ACARA, n.d.) refers specifically to nets as a form of representation:

> Connect three dimensional objects with their nets and other two-dimensional representations (ACMMG111)

This paper builds on our previous work which investigated students' schemes for anticipating whether given nets would or would not fold to form a target solid (Knight & Wright, 2014). Schemes are defined as observable action structures whereby an individual connects situations, actions, and anticipated results (Von Glasersfeld, 1989, 1998). Enhancement of schemes comes from anticipatory thought as well as physical enactment (Tzur et al., 2013). One focus for investigating students' schemes was their use of visual and analytic strategies.

Visual and Analytic Strategies

An established lens for viewing the strategies used by students on spatial tasks is to characterise those strategies as visual or analytic (Presmeg, 1986). Ramful, Ho, & Lowrie (2015) describe a visual strategy as one in which the student uses a visual image of an object. This definition is honest to many definitions of visualisation as mental transformation of images without a change in perspective (Hegarty & Waller, 2004). In the case of visualising if a net will work a student who imagines the physical folding of shapes is using a visual strategy. An analytic strategy references the properties of the geometric object, as might be evidenced if a student recognises that adjacent sides in a net must form an edge of the target solid or that adjacent faces of a cube must be orthogonal. These kinds of strategies are labelled analytic because they necessitate the dis-embedding of component parts from the whole and thinking with those elements (Owens, 2015). The acts of reconfiguring the whole and testing for validity also require other high order thought processes such as synthesis and evaluation.

The demarcation between visual and analytic strategies is far from clear. Presmeg (1992) described rationality and visualisation as 'intertwined' while Zazkis, Dubinsky, and

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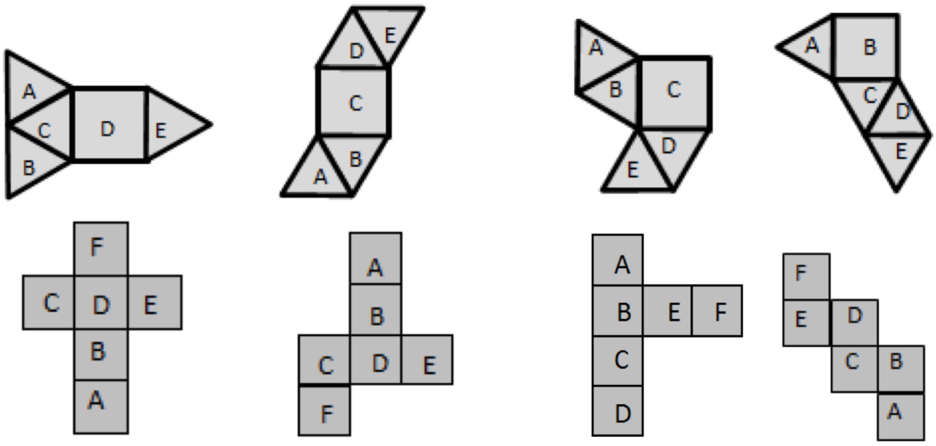
Dautermann (1996) proposed the VA model. In their model visualisation (V) and analysis (A) were connected through successive bootstrapping. Each act of visualisation was reflected on analytically resulting in new ways of seeing that informed further visualisation, and so forth. Arcavi (2003, p. 230) illustrated by examples that visualisation helps us to organise data in useful structures but “we see what we know”. The view of symbiotic interaction between strategies aligns with Duval’s seminal work on semiotic systems for mathematics (Duval, 1999, 2006). According to Duval mathematics is unique in that access to mathematical objects is through working with and connecting among representations. Nets are 2-Dimensional representations of 3-dimensional objects so much information about the object is not conveyed in single image. So acts of visualisation must be coordinated analytically with the mathematical objects they represent for the learner to gain ‘amplified cognition’ about the properties of that object. Meissner (2001) proposed that nets are procepts, an encapsulation of process (folding/unfolding) and anticipated result (properties of the target solid) embodied in symbolic form. Of interest in our work was how discrete visual and analytic strategies were identifiable in the schemes of students, and the relationships, if any, that existed between these strategies.

Method

Two interview protocols were developed, one each for the cube and square based pyramid. To find some information about their beginning concept images students were asked, “Do you know what a cube/pyramid is? What can you tell me about it?” A model of each solid made from polydrons™ was available, only if needed, to prompt the student. Each protocol had four nets shown individually on A5 sized cards (see Fig. 1). For each card students were asked, “Does this net fold to make a cube/pyramid?” After making a prediction the student was asked to “Explain how you know the net will/will not work.” Letters were used to label the faces in the net based on previous experience of students using ambiguous pronouns to refer to these features in their explanations. If a student claimed that a net would not work they were asked, “What could be changed so the net would work?” After making their predictions for all four nets students were given flat nets made of polydrons™ to check by folding.



Net A Net B Net C Net D

Figure 1: Nets for pyramid and cube

All the interviews were videoed. The interviewer completed a standardised recording sheet at the time. Without reference to the previous record the interviewer completed a new recording sheet from the video record to check for consistency. A research assistant also coded the video interviews independently to check for consistency. Thirty-six Year 6 students from an inner city Catholic school in central Melbourne were interviewed. The group made up all of the students in two classes within the school. In this paper we report in-depth on the strategies used by a sample of six students in their first interview. The sample group was created to achieve a range in the target solid used, gender, languages spoken at home (a measure of ethnicity), age, and raw score on NAPLAN Mathematics Year 5, a measure of global mathematics achievement (ACARA, 2013). Table 1 gives the composition of the sample group.

Table 1

Composition of sample group

Name	Net	Gender	Languages spoken at home	Age	NAPLAN raw score/40
Beth	pyramid	female	English, Tigray	11 years 11 months	21
Oprah	pyramid	female	English	11 years 1 month	31
Charlene	cube	female	English	11 years 3 months	23
Elijah	Cube	male	English, Italian	11 years 9 months	24
Geoff	pyramid	male	English	11 years 11 months	34
Jim	Cube	male	English, Italian	11 years 7 months	31

Results

Kozhevinikov, Kosslyn, and Shephard's (2005) classification of students as either object or spatial visualisers suggested that marked differences in descriptions of the target solid might predict success in the folding tasks. However, this was not the case. Jim, the poorest performer described a cube as having six square sides (altered to faces) and cited die and boxes as examples. His response had both object and spatial elements. Geoff, the most fluent performer, said of a square-based pyramid, "It stands up like most other 3D shapes. It's got triangles on the sides. It can be a square based pyramid. It has a point at the top." Similarly, his response contains reference to the objects appearance, "standing up", and to spatial properties, faces and an apex.

Assigning correctness to a particular student's response was complicated by the changes of mind, particularly by Charlotte and Beth, and operations that occurred during explanation. It was often the classic case of students' understanding not remaining still enough to assess it. For reporting the student's final decision was used though the records of deliberations provided rich data for investigating the interaction between visual and analytic strategies. Four students, Beth, Oprah, Charlene and Geoff correctly predicted all four nets though there were notable differences in certitude and reflection they needed to make the predictions. Elijah and Jim correctly predicted for two nets and one net respectively. Beth and Jim were credited with correct predictions when their explanations contained errors. NAPLAN scores did not associate with success on the nets tasks. Jim, a high scorer, did poorly on predicting nets while Charlene and Beth, both low scorers, predicted all nets correctly.

Co-gestures are commonly used by students in working with spatial tasks (Hostetter & Alibali, 2007) and are particularly associated with explanation. Three students, Beth, Charlene and Jim used gestures in forming their predictions but not consistently for all nets. For example, Elijah used finger folding on nets A and D but no gestures for nets B and C. There was no evidence of gesturing to predict by the other three students. However, all students used co-gestures to support their explanation as to why a given net would work or not work. Table 2 shows the gestures used in their explanations with (×n) used to represent multiple instances. Dynamic movements of fingers, palms and eyes signal imaging of movement and are indicative of visual strategies. However, these movements also have an analytic component. For example, folding of palms to an orthogonal position for a cube or at acute angles for a pyramid indicates a mapping to the relative position of faces. Static gestures were mainly used to either identify particular faces or to indicate the positional structure of faces. Positioning of palms in a static way to indicate the relative position of faces, orthogonal for a cube and meeting at a vertex for a pyramid, indicates a co-ordination between the net, the body as a semiotic register, and student's concept image for the target solid.

Table 2

Gestures used by students to explain their predictions

Name	Static gesture	Dynamic gesture
Beth	Finger locating (x4), Palms positioning	Fingers folding (x3), Hands rotating, Head movement, Rotating net card
Oprah	Finger locating (x 4)	Fingers folding (x 2), Hands rotating, Palms rotating
Charlene	Finger locating (x4), Palms positioning	Fingers folding (x4), Hands rotating, Palms folding (x2)
Elijah	Finger locating (x4), Hands closing	Fingers folding (x4), Palms folding, Hands rotating
Geoff	Finger locating (x2), Palms positioning (x2)	Fingers folding, Palms folding (x4), Eye movement
Jim	Finger pointing (x4)	Fingers folding (x4), Palms folding

The incorrect predictions of Jim and Elijah illustrated some factors that impact on students' ability to correctly predict folding a net. Below Jim described why Net B for the cube would not work. The left figure shows his first attempt, the right figure shows his second. During his reflection Jim changed the base from C to D. While he controlled a single orthogonal fold (F) and connected it to faces of his concept image this change of base required control of two consecutive orthogonal folds (L composite of FCD). He incorrectly believed F would form the top. Furthermore, Jim could not predict the location of A and B after folding.

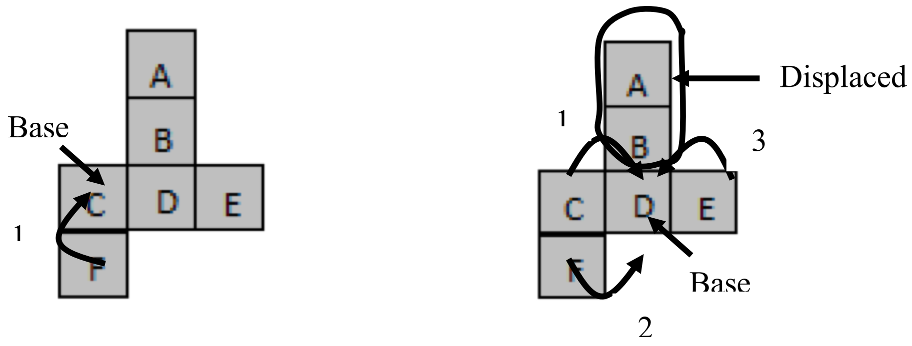


Figure 2. Jim's prediction for Net B

J: I don't think so. The way I was doing it I would put F folding up, facing me. C would be…(uncertain pause). D's the base, bottom. Then C would fold to…um...D. F would go on top. E would go on one side

I: Why won't it work?

J: If I think F goes on the top I don't know where these shapes would go (A and B), all over the place

Anticipating the result of folding was not a purely visual process as it required a mapping of shapes to corresponding positions of faces for the solid and positional relationships between those faces. Like Jim's "All over the place." Elijah used a statement of dislocation to describe why Net D for the cube would not work, "But A, B, and C would just be left. You can't really do anything." Elijah's discourse was that action was impossible because there was no way that

A, B and C could form the missing faces of the cube. His affordance for inaction shows no effort to co-ordinated folding, a visual strategy, with the positioning of faces, an analytic strategy.

Assigning the base was the first spatial to analytic connection made by students and affected the complexity of consequential folds. Strategic choice of base is known to influence memory demands and the success of students on net folding tasks (Owens, 2015). Elijah consistently made poor choices for the base square which necessitated complex folding sequences that he was unable to coordinate. For Net B he assigned B to be the base when D was an easier choice (see Fig. 3). He predicted that C and E would be parallel yet his statement that square F will "just go over" states the impossibility condition that F cannot form the top of the cube. He showed a chain of visual to analytic connections that ended when he could no longer coordinate the final connection he needed.

> E: I did two ways. First a made B the base, flipped A up to be one of the sides then kinda flipped these four (CDEF) up so D would be another side. These two (C and E) will go into B (uses palms to show parallel faces). You need a lid. The F will just go over.

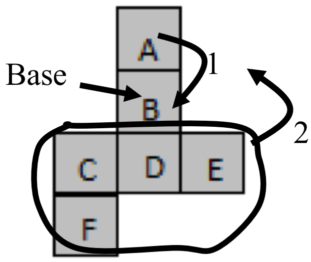


Figure 3. Elijah's choice of base and consequential folds

Statements of impossibility conditions were common in situations where students believed, correctly or incorrectly that a net would not work and took two forms, displacement of a face or faces, and overlaps of faces. Impossibility conditions might be considered analytical strategies. However, the rationale behind the statements revealed more than the statements themselves.

Beth, for whom English was a second language, struggled to express her thoughts. She requested and was given access to the polydron™ model of a pyramid. With the concrete model she enacted mappings of shapes to faces in a way that was not visible with the other students who carried out mappings to their concept image. She predicted that Net C was impossible owing to A needing to be in a different location (see Fig. 4). Her suggestion was partially correct in that a triangle was needed to connect to the right edge of C but she neglected the overlap o

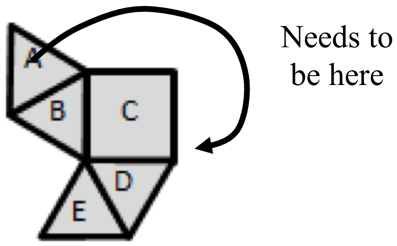


Figure 4. Beth's impossibility condition

Beth's statement that A needed relocating signalled an analytical strategy. If the square was sensibly assigned as the base it needed to be in contact with four triangles, one on each side.

However, she treated that property in isolation and did not check her assertion visually by imaging a folding process.

B: No I don't think this will work (rotates card back to original position). This square is stuck. You need four sides (pointing with finger gestures). The A and B go here but the A needs to be there (refer to diagram).

Oprah who was highly proficient on all four tasks also dismissed Net C for two reasons. While she used the same 'surround the square base' property as Beth, Olivia was able to mentally enact folding the net to establish missing and overlapping faces.

O: No. It won't. Because there's nothing to 'do' this side (pointing to right side of square)

I: Is there another reason?

O: And if you fold it the E and B will be together

Further evidence of the rich, iterative interplay between visual and analytic strategies was provided in situations of prolonged reflection. For example, Charlene who was finally successful on predicting all four nets, changed her mind several times (see Fig. 5). The interview showed her mapping between the result of folding actions and properties of her concept image for a cube, namely faces and their relative location.

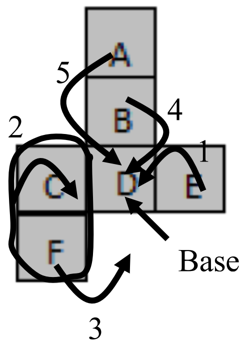


Figure 5. Charlene's folds for Net B

C: No. Because…wait let me think more. I say no because the F there can't do anything because they (C and E) fold up…and then, but it can!…I am so sorry. That (F) can fold around there because the C will be up there (Using palms to locate faces).

I: And you're saying that this (F) can fold around to be here (pointing) but what happens to these ones?

C: No it can't, Sorry. Unless you glue the A on top of the F.

I: As it is. If you were to cut it out, fold it up…

C: Then it wouldn't work because that there's (A) extra there. If A wasn't there it could fold up like that (using fingers to describe folding movements forming all faces but the top). Oh…Oh my God, it will work because A will become the top.

Charlotte's mind shifts contrasted with Geoff's responses that were fluent and confident. Geoff described Net B for a pyramid as "just like the other one (Net A)". He indicated that the location of triangle E was the only change and its new position "will still meet (sic)" the apex. There was recognition of isomorphism of structure in the two nets, a strongly analytic strategy. Geoff's description for Net D sounded like a visual enactment of the folding process. However, his fluency was founded in a trust that composites of shapes mapped to organised structures of faces for a pyramid. In particular, he knew that the 'boot' composite of B, C, D and E wrapped to form the base and three triangular faces of the pyramid. He possessed a trusted connection between action on the composite of shapes and the anticipated result of that action.

G: I reckon it would because A can come up and match the middle and C can come up. And these two (pointing to D and E) will be over. And D can go there (pointing) and E can wrap all the way around and come up as well.

There were several other examples of students development of trusted composites, sometimes after folding the polydron™ nets to check their predictions. For example, Ethan learned that an L shaped collection of squares formed half a cube. He said, "I didn't really think you could twist it like that." Composite thinking serves to reduce load on working memory.

Discussion

It is possible to discriminate students' strategies that are visual from those that are analytic. However, productive approaches used by these students suggested strong cycles of connection between visualising folding of nets and anticipating the structures formed. These structures were mapped analytically to concept images for the target solid. Cycles of visualisation and analysis were observed supporting the VA model of Zazkis, Dubinsky, and Dautermann (1996). Duval's (1999, 2006) theory of semiotic registers explains how 'amplified cognition' about the properties of target solids occurs by students connecting within and among representational registers.

This work suggests possibilities for further research. A larger sample of students might establish differences in schemes between competent and less competent predictors that, in turn, would inform ways to focus students' attention. Further work might also establish what features of nets in combination with students' selection of strategies contribute to task difficulty. The tasks we used placed students in receptive mode and more research is needed on students' productive mode with nets, as used in Piaget and Inhelder's seminal work (1956).

Instruction for anticipating nets should support students in connecting their visual strategies with properties of the target solid, the concept image. Further pedagogical research is needed to find approaches that support students to make these connections. These data suggest that successful teaching must involve students in cycles of mapping between action and concept through anticipation, and thinking with anticipated results to establish properties. Physical experience of folding nets alone, while beneficial, is unlikely by itself to promote strong spatial reasoning.

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