Teaching Fractions for Understanding: Addressing Interrelated Concepts

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The concept of fractions is perceived as one of the most difficult areas in school mathematics to learn and teach. The most frequently mentioned factors contributing to the complexity is fractions having five interrelated constructs: part-whole, ratio, operator, quotient, and measure. In this study, we used this framework to investigate the practices in a New Zealand Year 7 classroom. Video recordings and transcribed audio-recordings were analysed through the lenses of the five integrated concepts of fraction. The findings showed that students often initiated unexpected uses of fractions as quotient and as operator, drawing on part-whole understanding when solving fraction problems.

Fractions are notoriously difficult for students to learn and present ongoing pedagogical challenges to mathematics teachers (e.g., Behr, Lesh, Post, & Silver, 1983; Siemon et al., 2015). They are crucial, however, to students' future understanding of concepts such as proportional reasoning that are necessary not only for deeper mathematical understanding but also to support daily activities. These difficulties are often observed across all levels of education beginning from early primary years (e.g., Charalambous & Pitta-Pantazi, 2006; Empson & Levi, 2011; Gupta & Wilkerson, 2015). Different reasons have been identified for these difficulties, particularly in primary school. For example, fraction understanding is underpinned by larger mathematics cognitive processes including proportional reasoning and spatial reasoning (Moss & Case, 1999). In relation to having different notions of fractions, Hackenberg and Lee (2015) showed that limited understanding of particular aspects of the different meanings of fractions affects the ability of students to generalise and to work with fraction concepts. Similarly, Siemon et al. (2015) indicated that learning fractions is difficult because they are commonly used to represent a relationship between numbers rather than an absolute quantity.

Various studies considered the existence of interrelated fraction concepts as a major factor contributing to the difficulty of developing fraction understanding (e.g., Behr, Khoury, Harel, Post, & Lesh, 1997; Behr et al., 1983; Charalambous & Pitta-Pantazi, 2006; Siemon et al., 2015). Kieren (1976) was one of the earliest researchers to recommend that fractions be conceptualised as a set of interrelated constructs (ratio, operator, quotient, and measure) in teaching fractions for understanding. Behr et al. (1983) further extended Kieren's (1976) ideas of interrelated constructs of fraction and developed a theoretical model for learning by adding one additional construct (part-whole).

In this study, we used Behr et al.'s (1983) model of interrelated constructs of fractions to analyse a single lesson in a New Zealand middle school. Hence, this study is guided by the research question: Which constructs of fractions were reflected in the teacher's and students' discussion and use of language? The importance of this study lies in the use of Behr et al.'s (1983) model to analyse classroom interactions through language use.

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Background

Behr et al.'s (1983) theoretical model of interrelated concepts of fraction provides a way of considering pedagogical emphases. In the following section, we describe the five interrelated concepts of fraction and their classroom implications for teachers and students.

Part-Whole Concept

The part-whole construct of fractions is defined as a situation in which a continuous quantity or a set of discrete objects is partitioned into parts of equal size (Behr et al., 1983; Siemon et al., 2015). This representation is commonly used in the teaching of fraction concepts because it is assumed that students' initial intuitive experiences of fractions are derived from fair sharing (Siemon et al., 2015). The part-whole concept of fraction helps to answer the question "How much of an object or set is represented by the fraction symbol?" Although the part-whole concept of fraction is considered fundamental for developing an understanding of fraction concepts (Behr et al., 1997), it has limitations (Siemon et al., 2015). For example, the "out of" relationship between the whole and parts can only apply to proper fractions (4 out of 28 makes sense but 44 out of 28 does not). Therefore, addressing the other fraction concepts is important to develop a deep understanding of fractions.

The Ratio Concept

The concept of ratio is related to a comparison or relationship between two quantities in a given order rather than being a number by itself (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2006). The fraction notation 3/5 may also represent a ratio. For example, in a class of six boys and 10 girls, the ratio of boys to girls is 6 to 10, which is equivalent to 3/5. That is, for every three boys, there are five girls. The ratio interpretation of fractions does not involve the idea of partitioning and is hence conceptually different from the part-whole and quotient concept (Reys et al., 2012).

The Operator Concept

According to Charalambous and Pitta-Pantazi (2006), "the operator concept results from the combination of two multiplicative operations or as two discrete, but related functions that are applied consecutively" (p. 4). It is often indicative of multiplication (Behr et al., 1983; Siemon et al., 2015), particularly the interpretation characterized as "taking a part of the whole", such as one-quarter of a whole number. To master the operator concept of fractions, Charalambous and Pitta-Pantazi (2006) suggested that students could be engaged by multiplying or dividing fractions in a variety of ways (e.g., 3/4 should be interpreted either as 3 × [1/4 of a unit] or 1/4 × [3 units]).

The Quotient Concept

The quotient concept is fraction as division (Park, Güçler, & McCrory, 2013). The fraction 1/4 results from dividing 1 by 4. This interpretation of fractions is often ignored in classrooms (Park et al., 2013) despite providing a firm foundation for students to rename and compare fractions as decimals (Behr et al., 1983; Siemon et al., 2015). This construct also provides an opportunity for students to recognise that a fraction may have an infinite number of equivalent forms.

The Measure Concept

The measure concept of fraction can be interpreted as numbers that can be ordered on a number line. This notion is important for adding and subtracting fractions. For example, if two fractions with unlike denominators are interpreted as a measure (i.e., as distances from zero on the same scale), then they can be added or subtracted as measures only if they have the same units (Charalambous & Pitta-Pantazi, 2006; Siemon et al., 2015).

Generally, students must be comfortable with all of these interpretations of the fraction to have deep fractional understanding, and they must be able to do so without confusing whole number characteristics with fraction characteristics. In addition, understanding of fractions depends on gaining an understanding of each of these different meanings, as well as of their confluence (Behr et al., 1997).

Method

The Participants

As part of a larger study being conducted in Australia and New Zealand, 11 Year 7 students and a mathematics teacher participated in the study. According to the New Zealand Curriculum (Ministry of Education, 2007), students at this year level are expected to understand that the value of a fraction of an amount depends on both the fraction and the amount (for example, $1/3$ of $180 = 180 \div 3 = 60$), and to apply additive and multiplicative strategies flexibly to fractions, such as ratios. The students were from a very low socio-economic middle school in New Zealand that was participating in a local numeracy research project in which the class was divided for instruction, with about half the group working intensively with the teacher while the remaining students worked independently on set work. As part of that project, they were used to having visitors to their classroom, and to being video-recorded, and welcomed researchers. Following ethics approval, students were informed of the study and their consent was obtained.

Data Gathering and Analysis

A single hour-long lesson was video-recorded and analysed using the interrelated theoretical model of fraction constructs (Behr et al., 1983) to analyse the dialogue between the teacher and students in the classroom context. The analysis focused on identifying each of the five different fraction concepts as the lesson unfolded and the teacher responded to students' comments and ideas. The description for each fraction construct and related concepts are shown in Table 1, which was used to guide the analysis process.

The observation data were coded by watching the video-recorded lesson and using the audio transcript. Based on the descriptions shown in Table 1, notes were made when evidence was observed, and a frequency table was prepared to indicate concepts of fraction reflected in the teacher's and students' discussion and use of language. Barron and Engle (2007) suggested that video-audio transcription coding should be iteratively revised until the transcripts eventually provide a reliable record of what the researchers view as the most relevant aspects of the video for their research questions. Consistent with the advice of Barron and Engle (2007), the analysis emphasised the identification of fraction concepts as described in Table 1 through the language used by the teacher and students as the class worked on solving fraction problems.

Table 1
Fraction Constructs, Relevant Concepts, and Assessment Strategies

Fraction constructs	Relevant description/concepts
Part-whole	The process of partitioning, parts are equal. Relationship between the whole and the parts
Ratio	Concept of equivalence, comparison or relationship between two quantities
Operator	Division, or multiplication operation on fractions, taking a part of the whole
Quotient	Addition or division operation on fractions. A single rational number derived from dividing the nominator by the denominator
Measure	Order or identify a number represented by a certain point on the number line

The Classroom Context

After introducing the lesson topic, students were given two consecutive tasks to complete in groups of three or four. First, they were asked to sort a set of the fraction strips in ascending or descending order. The teacher posed follow-up questions while students were sorting out the fraction strips, such as "What can you tell me about ordering?", "Why is it a smaller fraction?", and "What is the relationship between the bottom and the top fraction?" The top and the bottom could be confusing terms to use; however, the students were able to later understand the terms as nominator and denominator of a fraction.

After completing the first task, students in groups were asked to choose and solve one of three problems. All students had to be prepared to show their group's solution on the board and to answer any questions asked by the other students. The context of the problem was a visit to McDonalds following a sports game. The teacher ensured that the students were familiar with this context before asking individual students to read the fractions questions. The three questions were of increasing difficulty and complexity:

1. 4 burgers were ordered and half of these given to another person.
2. 16 burgers ordered and one-quarter given to another person.
3. 40 burgers ordered and five-eighths given to another person.

The third question was clearly much more complex and intended only for competent students, although any of the groups could have attempted it.

Results

Table 2 summarises the frequency of the observed concepts of fraction reflected in the teacher's and students' discussion, and use of language. Illustrative examples are taken from the teacher's and students' discussions while completing the two tasks.

As shown in Table 2, the most frequently observed fraction concept reflected in the teacher's and students' discussion, and use of language was part – whole ($F = 40$) whereas measure ($F = 4$) was the least.

Based on the requirements of the first task, all the students were able to order the fraction strips. As they did so, the students (S) discussed the questions posed by the teacher

(T). The conversation between the teacher and the students shown below is an illustrative example of the part-whole concept of fractions.

T. So, can you make a connection with this one right at the top and the one – and maybe one of the others? Anyone?
S: That's one whole.
T: That's one whole. And what's down here?
Students provide different answers.
T: Yes. One whole is ten tenths or twelve, what?

Table 2
Frequency of Observed Fraction Concepts

Fraction concept	Frequency	Example
Part-whole	40	Identifying the relationship between the one whole fraction strip and the parts (1/10, 1/8 etc.); partitioning the whole into a given number of parts in Task 1
Ratio	8	The concept of equivalence was discussed while identifying the whole and parts, such as "1 whole is 10 tenths or 12 twelfths"
Operator	32	Finding 1/4 of 16 is similar to multiplying 1/4 by 16
Quotient	10	The students understand that the value of a fraction of an amount depends on both the fraction and the amount, for example, 1/2 of 4 hamburgers is 2
Measure	4	Arranging fraction strips from biggest to smallest fractions, and representing one-quarter as 1/4

Another group was engaged with the concept of equivalence (ratio), connecting the denominator of a fraction with the size of the fraction strips, illustrated by the following conversation. In this conversation, the teacher intended to demonstrate, for example, two smaller strips of 1/4 are equivalent to 1/2 or the ratio of 2/4 is equivalent to the ratio 1/2.

T: What is the relationship between the size of the square and the bottom number?
S1: The pieces get smaller.
T: Right. And why do you think that's happening?
S1: Because there's [indecipherable] there's more on that one.
T: So, talk about the pieces are getting smaller and the numbers are increasing. So, what do we mean? Try and make a connection there.
S2: The bottom number.
T: The bottom number. So, we're thinking about the bottom number. Does that bottom number have a name?
S3: The denominator.
T: Denominator. Yes. Alright. So. Interesting. So, I can gather from this little discussion that we've had that some people know about numerator and denominator. Some people have recognised that the pieces are getting smaller and the numbers are increasing.

These discussions show that the process of partitioning, the relationship between part and whole, and the concept of equivalence and equivalent fractions were addressed within a single task of organising fraction strips by size.

After finishing the first task, the student groups were engaged in solving their chosen problem in the second task. Each group presented their answer to the rest of the class. The first group presented their solution for the first problem on the board as shown in Figure 1.

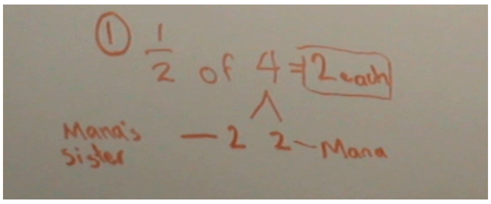

Figure 1. Students' work showing the concept of operator and quotient.

Watching the recorded video and as shown in Figure 3, the students showed that they understood that the value of a fraction of an amount depended on both the fraction and the amount and recognised that one-half of four is equivalent to two, implicitly using both the operator concept and the quotient idea. The operator concept of fraction was also observed in the work of another group (Figure 2). The teacher asked what operation to use. The students redefined one-quarter of 16 as multiplying 1/4 by 16. In addition, they were able to define multiplication as repeated addition to check their answer (16 = 4 + 4 + 4 + 4), making connections to prior mathematical knowledge.

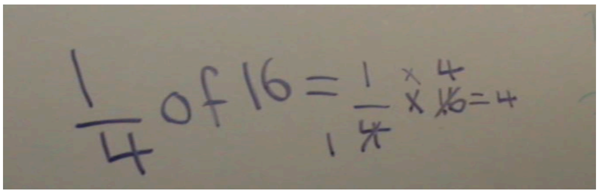

Figure 2. The operator concept of fraction.

The quotient concept was also observed using division, as in this conversation.

T: So, you're imagining that you've got 40 hamburgers. Yes. Okay. And what's the connection to the eight?
S. You're dividing.
T: Okay. Dividing what?
S: To five groups.

In this example, the students were showing the connections between 40, eight, and the result of five groups; that is, dividing 40 by eight resulted in five. They did not use the language of sharing, which is a part-whole notion.

The measure concept was also seen but to a lesser extent and only through the teacher's dialogue, whereas the other ideas were often initiated by the students. For example,

T: Four. What does four quarters look like as a number? Well done. So, there's a quarter in each part. What does four quarters look like? So, this is what a quarter looks like. What does four quarters look like as a number?
S: A whole.
T: It looks like a whole. Yeah. But what does it look like as a number?

This dialogue between students and the teacher indicated the concept of the part-whole relationship that is four quarters form one whole and implicitly considered the idea of a fraction as a number that is the measure concept.

Discussion and Conclusion

Given the nature of the problems posed by the teacher, it is not surprising that the measure concept was only tangentially observed. Of interest, however, was that the students initiated many of the concepts, reinforced by the teacher's questioning. Students seemed to move seamlessly between the ideas, sometimes implicitly drawing on different ideas within the same sentence. The lesson started with a reinforcement of the part-whole concept as commonly practiced in teaching fractions concepts because it is assumed that children's initial intuitive experiences of fraction are derived from fair sharing (Siemon et al., 2015; Strother et al., 2016). In this study, however, the teacher was able to draw out students' understanding of fraction concepts by addressing the other interrelated ideas (ratio, operator, and quotient) through engaging students with practical problems and using manipulatives, and an emphasis on discussion and students explaining their thinking. Similar to other studies (e.g., Charalambous & Pitta-Pantazi, 2006; Siemon et al., 2015), the present study showed that the most frequently observed fraction concept reflected in the teacher's and students' discussion, and use of language, was part-whole whereas measure was the least. This could be reflected in the students' competency with fraction concepts. For example, Charalambous and Pitta-Pantazi (2006) showed that students were successful in tasks related to the part-whole construct and least competent with tasks corresponding to the measure construct.

Fraction concepts are often taught using procedures and memorisation rather than having students develop their own understanding (Siemon et al., 2015). The use of manipulatives in teaching fractions with students working in small groups, and discussion and questioning among students explicitly encouraged, allowed the students to draw on other knowledge, such as repeated addition and "tables" knowledge, and relate this to fraction understanding. Students may, however, have been able to better see the links to ratio and measurement concepts of fraction if they were provided with additional manipulatives such as a thin strip of paper. As suggested by Reys et al. (2012), the strip of paper could be folded into halves, quarters, and so on, and later, students could use length partitioning to represent fractions as points on a number line.

As Tobias (2013) has shown, if the language used by the teacher is incorrect or confusing to explain fraction concepts, students may continue to use the teacher's incorrect language to describe fractions, and not understand the concepts clearly. For example, questions using language such as "bottom" and "top" number could be potentially limiting for students because "bottom" and "top" don't clearly describe fraction concepts.

This small-scale study showed that the use of interrelated fraction concepts in conversation might have implications for mathematics teachers' pedagogical and assessment strategies. For students to have a good understanding of fraction concepts, teachers need to include different constructs of fraction concepts in their pedagogical and assessment approaches (Siemon et al., 2015). Such use needs to be deliberate and focussed, whereas, in this study, the usage of different concepts arose informally through classroom discussion. Charalambous and Pitta-Pantazi (2006) used the model as a reference point to investigate students' notions of the different constructs of fractions through administrating a test. Other studies used Behr et al.'s (1983) model to analyse students understanding of one element of fraction concepts (e.g., Empson & Levi, 2011; Mitchell & Horne, 2009). This study has shown that the same model can be applied to normal classroom discourse.

Overall, this small-scale study showed that although students' intuitive understanding of fractions as part-whole provided important prior knowledge, they also used language consistent with other concepts of fraction, such as ratio, operator, and quotient, with teacher encouragement. Perhaps a next step is to make these concepts explicit to the students by developing the language use of the more unfamiliar terminology. In this study, similar to Kieren's (1976) suggestion, the ratio, operator, and quotient concepts of fraction were often reflected in part-whole contexts during the dialogue between the teacher and students. However, because of the limited scope of the tasks that the students completed, there were limited opportunities for students to use measurement concepts. The study could have more to say on the measurement concepts if the students were engaged in more diverse activities.

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